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ERRATUM.

Vol. vii., p. 123, line 18, for $\varepsilon^{-\frac{m}{\log. q} \cdot q^{m}}$, read $\varepsilon^{-\frac{m}{\log. q}}$.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Memoir of the late Benjamin Gompertz, F.R.S., F.R.A.S., &c. By MARCUS N. ADLER, M.A., *Fellow of the Institute of Actuaries and of the Statistical Society.*

[Read before the Institute, 18th December, 1865.]

IN the demise of Benjamin Gompertz the scientific world has recently had to deplore the loss of a man, to whose labours the student of both pure and applied mathematics, but especially the actuary, is deeply indebted. The death of this distinguished *savant,* who may be considered as having been, in his later years, a connecting link between the past and present generation, should not be passed over in silence. It is a duty we owe as much to the man as to society, to trace, whilst his memory is still fresh, the chief events of his life, and to notice his principal contributions to science.

Benjamin Gompertz, a member of the Jewish community, was descended from a family which long held a distinguished position in Holland. His grandfather on the mother's side, Benjamin Cohen by name, was on intimate terms with William, Prince of Orange; and it is related that during a revolution in the middle of the last century the Dutch Stadtholder found a ready asylum at the magnificent mansion his friend had erected near a place called Amersfort. Cohen zealously patronized the arts and sciences, and was himself a mathematician of some pretension.

A curious translation of Euclid into Hebrew, and a little treatise on dialling and gnomonic projection, which were composed for Cohen's use, may even now be seen in some of our libraries.

The father of Benjamin Gompertz was a successful diamond merchant, a pursuit in which the Dutch chiefly seem to excel. Even at the present day the art of diamond cutting in Europe seems to be confined to Holland; and a visit to any of the factories there may convince the reader that the management of such an establishment requires abilities of no mean order.

The brothers of Mr. Benjamin Gompertz had little taste for a commercial life, but shared his predilection for literary pursuits. Barnet, the eldest, was an eminent amateur musician.

His brother Isaac wrote poems, which at the time were much admired.*

Another brother, Ephraim, who is still living, is an original mathematician. It is only to be regretted that so few of his investigations have been published.† Three able pamphlets written by him many years ago, on the gold question, are deserving of attention even at the present day, when we find that such discordant opinions are held on the subject by our leading thinkers.‡

Lewis, the youngest brother, was the founder of the Animals' Friend Society, and entertained some strange ideas on the relation of the brute creation to man. He published several works,|| in which he endeavoured to prove that it was not only unlawful to kill an animal, but to turn it to any use that was not directly beneficial to the animal itself. Accordingly he did not object to drink milk, or to wear cloth, because milking the cow, when not robbing the calf thereby, or shearing the sheep, was for the good of those animals; but to eat meat or to drive in a coach was contrary to his doctrine. He always maintained, and this long before railways or steam ploughs were introduced, that machinery might be made perfectly subservient to our wants. He certainly lived to see the verification of a portion of his theory. We are assured that in his will he left handsome annuities for the maintenance of some of his pet animals. It may be added, that he was of a very inventive turn of mind, and the author of several contributions to mechanical science.

---

* *Vide* the first number of *The Liberal*, edited by Leigh Hunt. Among the chief poems of Isaac Gompertz may be specified "The Modern Antique," "Time, or Light and Shade," "Devon," &c.

† *Vide* T. Baker's *Formulæ, Rules, and Examples.* Weale, 1862.

‡ *Theoretical Discourse on the Nature and Property of Money.* Richardson, 1820. *An Essay on Currency;* 1829. *An Attempt at an Analysis of the Currency and the Merits of the Bank of England;* 1830.

|| *Moral Inquiries on the Situation of Man and Brutes.* London, 1824. Also Fragment in defence of animals.

Benjamin Gompertz was born on the 5th of March, 1779, at Bury Street, in the city of London. He did not enjoy the benefit of a collegiate education, and on the whole he may be considered as self-taught.

So great was his thirst for knowledge when a mere boy, that frequently, when his parents had removed all candles to prevent him from injuring his health by studying too late at night, he stole out into the garden, and pursued his investigations by moonlight. He was wonderfully familiar with the writings of the English and French mathematicians of the last century; but Newton, Maclaurin, and Emerson, were his favourite authors. The admiration he entertained for Newton was so great, that he clung to his Fluxional language and notation long after these had been abandoned by other English writers. It may be interesting to know the grounds upon which he based his great preference for the fluxional to what he calls "the furtive" notation used on the continent, and now much used in England.*

"I call the differential notation furtive, on I think a moral ground, and also on the ground of its introducing an interruption and an inconvenience in practice. The moral ground is, that it appears to give Leibnitz a greater claim to originality, to the prejudice of Newton, than I think he is justly entitled to. And the other ground is, it steals from the alphabet a letter—and one which it is most convenient to retain, in order to keep up the regular order of notation—to use it for a purpose of different intent to that for which it was originally used, and may introduce confusion. And with respect to the superior advantage of the fluxional calculus over the differential calculus, I observe that if x and y be rectangular co-ordinates of a curve in a plane, and z the length of the curve from a given point in it to the point of which x and y are co-ordinates, the fluxional calculus gives $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, which is strictly true, and may be proved to be so without the introduction of infinitely small quantities: but the differential calculus gives $dz=\sqrt{\overline{dx^2}|+\overline{dy}|^2}$, true only on consideration of infinitely small quantities; and even with that consideration it cannot be proved luminously to be true, because $\sqrt{\overline{dx}|^2+\overline{dy}|^2}$ only expresses the length of the chord of an infinitely small arc, and not of the arc itself, as they have no part common with each other, but at the points of intersection. . . . . . And similar observations may be made with respect to the luminous character of the fluxional analysis when compared with the differential analysis, in the application of it to physical subjects. Thus if f be a force acting on a body, $\dot{v}$ the velocity which is generated by its momentary impulse—that is to say, its single impulse, by which is meant the finite space the body would describe in consequence of it in the finite time $\dot{t}$, but not the variable portions of space it would describe if that force were considered to be an infinitely small action, as it were continually active during infinitely small portions of time; the fluxional calculus gives $f\dot{t}=\dot{v}$, and is correctly true, however large $\dot{v}$ and $\dot{t}$ are. But the differential analysis gives $f.dt=dv$, which is correct only if dt and dv are infinitely small, and is then only to be considered so in virtue of the hypothesis that infinitely small quantities of the second and higher degree may be omitted."

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* *Philosophical Transactions*, Part I., 1862, p. 513.

We are not about to enter into that long-vexed dispute as to the relative claims of Newton and Leibnitz. Posterity has now decided that Leibnitz was an independent discoverer and the first promulgator of the calculus;* and the general adoption of his mode of analysis, as well as of his notation, shows that it has not been found so inconvenient in practice. On the other hand, we venture to think that the doctrine of fluxions has the advantage of perspicuity of reasoning and clearness of illustration. Indeed it is a matter of surprise that whereas Newton's method was the only one acknowleged in England during the last century, its use should since then have been discarded up to within the last few years. A change seems to be now creeping in, so far at least as the Newtonian mode of reasoning is concerned; and we are inclined to think that Mr. Gompertz, by his steadfast advocacy of the claims of our illustrious Newton, deserves some of the credit for the reaction which has set in. In the paper from which we have already quoted Mr. Gompertz explains his own peculiar notation of partial fluxions or differentials; he also exhibits a mode of indicating the common or Napierian logarithms or anti-logarithms by a new foot-index of crosier shape.† Again, his embodiment of zeros will be found quite a boon in calculations where large numbers or very small fractions occur. Where but a few significant figures are of importance, he puts these down followed or preceded by the actual number of cyphers placed in a ring. Thus, 898000000 or ·00000000763 he denotes by 898⑥ and ⑧763 respectively. All these matters are worthy of attention, and it is but to be regretted that they are explained in a magazine that is not easy of access to the student of elementary subjects.

Mr. Gompertz whilst still a youth, in fact as early as the year 1798, took a prominent part among the mathematicians who proposed and answered the ingenious questions contained in the *Gentleman's Mathematical Companion*. Scientific men of high repute, such as Ivory, Nicholson, Griffith Davies, &c., contributed to this publication. The reader will at once be impressed with the great analytical powers possessed by Mr. Gompertz from the fact that, from the years 1812 to 1822 (when his want of leisure prevented him from continuing his contributions), he gained, without a single exception, the annual prize for the best solutions of the prize questions.

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* *Vide* article "Commercium Epistolicum," in the *Penny Cyclopædia*.

† He uses $\underset{\lrcorner}{q}$ for the common logarithm, $\underset{\llcorner}{q}$ for the Napierian logarithm of a number; also $\underset{\urcorner}{q}$ for the common anti-logarithm, and $\underset{\ulcorner}{q}$ for the Napierian anti-logarithm.

We will here only advert to a problem solved by Mr. Gompertz, in the *Companion* for the year 1822, to show how practically useful some of those inquiries were.

> The solidities of many solids can be found, provided the dimensions of the middle section of each particular solid is given. If A and B are the areas of the ends of the solid, C the area of a section parallel to and equidistant from the ends, and L the length or distance between the ends,
>
> $$\text{the solidity } S = \tfrac{1}{6}\{A + B + 4C\} \times L.$$
>
> The solidities of the wedge, the prismoid, the pyramids, and their frustums, also of the whole or a segment or any portion of the whole contained between two parallel planes perpendicular to the axis of a sphere, ellipsoid, paraboloid, and of a hyperboloid, may be found by this general formula.

Mr. Gompertz was intended by his father to follow a purely mercantile profession. In compliance with this intention he made his start in life as a member of the Stock Exchange, and continued as such almost to the time that he undertook the management of an Insurance Office.

In the year 1810 he married Miss Abigail Montefiore, the sister of the well-known philanthropist Sir Moses Montefiore, Bart. Their union was blessed with one son, who died young; and two daughters, both married, who survive him.

A commercial life was not to the taste of Mr. Gompertz, and he was glad to retire when he could from the noise of 'Change Alley to the calmness and quiet of his study, or to the instructive discussions at the different learned Societies of which he was a member. In this respect he much resembled a man for whom he entertained the warmest friendship, and who also made his start in life as a member of the Stock Exchange—the late Francis Baily. Among other distinguished men with whom Mr. Gompertz was on terms of intimacy we may mention the late erudite Mark Noble, Sir John F. W. Herschel, Sir Jas. South, Mr. Babbage, &c.

The first learned Society that Mr. Gompertz joined was the Mathematical, whose members met at Crispin Street, Spitalfields. The object of this Society, founded so far back as 1717, was to introduce to the middle classes studies which till then had been confined to the libraries of the learned or the investigation of the opulent. Their valuable library and rich collection of instruments offered great facilities to the student, and a constant interchange of ideas was fostered by frequent discussions and occasional lectures. The originality displayed by Mr. Gompertz, and his thorough acquaintance with all branches of pure and applied mathematics, soon gave him a prominent place among the members; and his urbanity secured for him their good will, so that eventually he became the President of the Society. Subsequently, when the number of members had dwindled down from 81 to the square root of that number, it amalgamated with the Astronomical Society.

Mr. Gompertz communicated his first original paper of importance to the Royal Society, in 1806, through the then Astronomer Royal, Rev. Dr. Maskelyne. It treated on the application of the method of differences to a species of series whose sums were obtained, by the help of impossible quantities, by Mr. Landen in his Memoirs. The method given is simple, somewhat akin to that taught by Euler in his *Institutiones Calculi Integralis.* The Scholium is perhaps the most interesting part of the paper, and discusses those particular cases which cannot be summed by the ordinary processes pointed out by Landen or by the author himself.

The next important subject that engaged Mr. Gompertz's attention was an investigation into the theory of imaginary quantities. He was anxious to communicate his results to the Royal Society, who seemed unwilling to give insertion to them in their *Transactions.* One day, meeting Dr. Wollaston, who was Secretary to the Royal Society at that time, Mr. Gompertz asked him the reason of their refusal. "The fact is," replied Wollaston, "your papers are too profound for us, no one would be able to comprehend them." Such opinions were flattering in one respect, though rather disheartening in another.

Mr. Gompertz, was, however, not deterred, and eventually published, at his own expense, a series of original tracts on mathematical subjects. Next to the papers on his law of mortality, he considered these his most successful essays.

The first of these, treating on imaginary quantities, dedicated to his distinguished pupil Miss Lousada, was published in the year 1817. This branch of mathematics has made wonderful strides within the last thirty years; still it may be interesting to know how the matter was viewed at the time we are speaking of. In the introduction Mr. Gompertz says:—

> "Immediately after the invention of methods for the discovery of the roots of an equation above the first degree, algebraic expressions must have

occurred, which seemed to require the extraction of the square root of a negative quantity, the impracticability of which could not have failed to throw much embarrassment in the way of the early promoters of algebraic knowledge. Ages of study on this subject have pointed out beauties where deformity seemed only to reign, and the reiterated attempts of analysts to unriddle the secrets of science have been rewarded by the discovery that those very expressions termed impossible or imaginary quantities, which were sterile in appearance, were the most powerful instruments we could possess. Still the method of their operation is far from being universally known, and the truths of their results are by no means generally acknowledged to be legitimately obtained, and would by many be wholly rejected in the absence of other demonstrations."

After pointing out what he conceived to be the original steps which led to the introduction of imaginary quantities in algebra, and showing what an assumed law of continuity ought to lead to in the operation, he advances the following fundamental "position":—

"That wherever the operation by imaginary expressions can be used, the propriety may be explained from the capability of one arbitrary quantity or more being introduced into the expressions, which are imaginary previously to the said arbitrary quantity or quantities being introduced, so as to render them real without altering the truth they are meant to express; and that, in consequence, the operation will proceed on real quantity; the introduced arbitrary quantity or quantities necessary to render the first steps of the reasoning arguments on real quantity, vanishing at the conclusion; whence it will follow that the non-introduction of such can produce nothing wrong."

Thus, in operating with $\sqrt{-1}$, we are to operate as we would with $\sqrt{\rho-1}$, and afterwards expunge ρ; or, in short, suppose ρ to be there without inserting it. Mr. Gompertz calls expressions containing arbitrary quantities, such as those above referred to, *porismatic* expressions, and devotes a voluminous Tract, published in the year 1850, to a most lucid discussion of different porisms. He applies his analysis to a solution of a number of geometrical problems, more especially to a class of questions connected with *loci*, or, as he terms them, *local porisms*. Reverting to the subject of imaginary quantities, Mr. Gompertz successfully applied his method to the solution and interpretation of numerous questions in which the square root of a negative quantity plays an important part, such as the irreducible case of cubic equations, the exponential forms of the trigonometrical functions, the solution of linear differential equations, &c.

In the second Tract, published soon after the first, and dedicated to his brother Ephraim, he gives a geometrical interpretation to imaginary quantities, by having recourse to what he styles "functional projection." We regret we cannot give here a sufficiently elementary description of the method. It is highly ingenious, though it must be admitted it is not so simple as the geometrical interpretation generally adopted at present, for a full account of which the reader is referred to Professor De Morgan's *Double Algebra*, and sundry papers in the *Cambridge Philosophical Transactions*. However, to convince the reader of the effectiveness of Mr. Gompertz's method, we would call attention to the prize question proposed by him in the *Gentleman's Mathematical Companion* for 1820:—"The sides of a pentagon are respectively as 1, 2, 3, 4, 5, and the three angles subtending the largest side are equal, what are those angles?" This will be found no easy problem, but by the principles of functional projection the question is solved without difficulty. The angles are found to be 114° 58′ nearly.

That all these labours were duly estimated by the scientific world is shown from the fact that on the 29th January, 1819, Mr. Gompertz was elected a Fellow of the Royal Society. He continued a prominent member of that illustrious body for above 46 years. Although it is rather anticipating matters, we may here state that he was elected a member of the Council on the 30th November, 1832, and served the Royal Society in that capacity for several years.

In the year 1820 falls the foundation of the Astronomical Society, which, notwithstanding the opposition it at first met with at the hands of the President of the Royal Society (who was apprehensive lest his own Institution might thereby suffer), soon took a most prominent position among the leading Societies of the kingdom. Mr. Gompertz may be considered one of its founders, or at all events its warm supporter from the day of its foundation. He was elected on the Council on the 9th of February, 1821, and he continued a committee-man for nearly 10 years. Many of the most valuable contributions to the Society were committed to his care, and not a few of those so entrusted to him he made more complete by supplying them with explanatory notes.*

His first contribution of importance which was placed before the Astronomical Society, was "On the Theory of astronomical instruments." He presents us therein with a method by which due allowances and corrections can be made, with great accuracy, for certain defects in the instrument.

---

* See Littrow's paper "On the correction of the transit instrument": *Astronomical Transactions*, vol. i., p. 275. Also Kreil's paper "On the use of the equatorial," vol. iv., p. 501.

Thus, if with the intention of forming and fixing a transit instrument, we form an instrument of which the axis of motion is not scrupulously fixed perpendicularly to the meridian, nor even parallel to the horizon, nor the line of collimation of the telescope scrupulously perpendicular to the axis of motion, nor the altitude limb perpendicular to the axis of motion, the value of the portions of the altitude limb may be obtained by what he styles "inverse construction," and the theory of the instrument laid down such, that, with certain provisos, the instrument with all its defects may answer the purpose both of a well-constructed transit instrument and of a mural quadrant. A theoretical method for the accurate adjustment of Hadley's sextant and kindred instruments forms the substance of two papers and of a supplement which was read the 12th December, 1823.

Investigations on the aberration of light seem to have next invited the attention of Mr. Gompertz. A passage which occurs in a paper on this subject, read the 14th June, 1822, gives such a vivid expression of the enthusiasm with which he followed scientific inquiries, that we cannot forbear from quoting it:—"In the acquirement of the sciences there is, besides the pleasure arising from the acquirement of knowledge of practical utility, a peculiar charm bestowed by the reasoning faculty in a well-directed pursuit of facts; and though the results shown by the arguments are frequently considered the only objects of value by the unlearned, the man of absolutely scientific ardour will often, whilst he is enraptured with the argument, have not the least interest for the object for which his argument was instituted." In the course of the essay, several of the problems on aberration which are treated of in Biot's *Astronomy* are arrived at by the author in a less symbolical manner, and are made to apply not only to the solar system but also to the fixed stars.

In another paper, read before the Astronomical Society in 1824, he explained a new instrument, appropriately called, by Mr. Francis Baily, the differential sextant, for accurately measuring the angular distances of any two celestial phenomena caused by such varying circumstances as produce but small alterations, such as refraction, parallax, or aberration.

Another paper, published in the *Transactions of the Astronomical Society*, is one on the convertible pendulum, for the discovery of which the world is so much indebted to Captain Kater. The property of the pendulum, that if the centre of oscillation relative to any proposed centre of suspension be made the centre of suspension, the proposed centre of suspension will become the centre of oscillation, can only be received as true if the axes of motion or knife edges are in the two cases parallel to each other; if the supports of the axis of vibration be not horizontal, the length of the pendulum performing the vibration in a given time will be shorter than if the pendulum vibrates (in the same time) on a horizontal support in the proportion of the cosine of inclination to the horizon to the radius. Mr. Gompertz enters more particularly into the solution of this problem.

One of the most practically important objects to which Mr. Gompertz's attention was drawn about this time, was the reduction of apparent to mean places of the fixed stars. To quote the words of Sir J. F. W. Herschel, in his excellent memoir of the late Francis Baily,* "It seems almost astonishing that these computations, which lie at the root of all astronomy, and without which no result can be arrived at, and no practical observer can advance a single step, should have remained up to so late a period as the twentieth year of the nineteenth century in the loose, irregular, and troublesome state which was actually the case; and that not from their theory being ill understood, but from their practice not having been systematized. Each of the uranographical corrections had to be separately computed by its own peculiar tables, and with coefficients on whose magnitude no two astronomers agreed. The latter evil, indeed, might be tolerated at a time when the tenth of a second of space was not considered of so much consequence as at present; but the calculations were formidable and onerous in the extreme to private astronomers, whatever they might be rendered in public establishments by habit and the use of auxiliary tables. So far as the fundamental stars were concerned, the subject had for some time attracted attention, and had begun to receive its proper remedy by the publication, by Professor Schumacher, in Denmark, of their apparent places for every tenth day; and by the laudable exertions of Sir James South, in our own country, who for some years prepared and circulated similar tables for every day. But sidereal astronomy for stars, beyond the bounds of the favoured list, might be almost said to have been interdicted to the private astronomer, owing to the excessive irksomeness of the calculations."

Messrs. Baily and Gompertz perceiving this want, proceeded to supply it. The subject was investigated generally, and a method devised for arranging the terms of the corrections for aberration and solar and lunar precession. Some of the tables had already been computed, when they heard of Bessel's labours in the same field. Finding that that astronomer had proceeded upon a similar principle, but that besides the other corrections he had taken in that of refraction, proper motion, &c., Messrs. Baily and Gompertz willingly gave way. Certainly nothing more perfect than the *Fundamenta Astronomiæ*, by Bessel, could have been desired. However, Baily and Gompertz's work was of some use. The complete Catalogue of Stars of the Royal Astronomical Society may be looked upon as one of the united labours of these men.

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* *Vide* Monthly Notices of the Royal Astronomical Society for November 8, 1844.

Although the last published paper by Mr. Gompertz on astronomy dates so far back as the year 1829, he continued to turn his attention to this branch of science from time to time.

During the last years of his life he was engaged in investigations on meteors, shooting stars, comets, &c., but he had to lay them aside to enable him to complete a paper for the Royal Society on life contingencies.

We must now advert to the labours undertaken by the subject of our Memoir in the field of life assurance.

On the 29th June, 1820, he presented to the Royal Society his sketch of an analysis and notation applicable to the estimation of the value of life contingencies. In the introduction he forcibly dwelt upon the desirability that Assurance Companies should promulgate the information and experience they individually acquired, with a view to be guided by more truthful tables than were at that time acted upon. He expressed the hope, that actuaries of different Societies would, by their mathematical skill, collect for the common good of all, from multiplied resources, that which they could not obtain from a less general observation.

All this will, no doubt, be considered very trite at the present day; but we must recollect that the science of life insurance was at that time in its infancy, and that it required continual urging, before the Government could be induced to ascertain the mortality of its annuitants, and the authorities of the Equitable Society that of their assured.

As regards the latter establishment, it may be said, that Mr. Gompertz was the first to make known, with any degree of accuracy, the mortality that prevailed amongst its members. Mr. Morgan, in the sixth edition of Dr. Price's work on reversionary payments, compares the mortality experienced at the different decennia of life in the Equitable Society with that deduced from the Northampton tables. By the help of a lemma,* and the application of his law of mortality, to which we shall refer further on, Mr. Gompertz constructed a complete table of mortality, starting from age 10, which tallies very closely with that deduced by Mr. Griffith Davies. Now, although the observations of both these gentlemen on this head were made public about the same time, viz., the year 1825; yet, on reference to Mr. Gompertz's original manuscripts, it will be seen, that he framed the table in its more important particulars prior to October, 1820.

In Tract I. Mr. Gompertz establishes a system of notation which is as ingenious as it is exhaustive, though it must be admitted, that it lacks that compendious character which would have recommended it for more general adoption.

By means of his methods of summation he determines more accurately than by the common modes of approximation the value of incomes dependent upon a number of joint lives. He then applies the method of fluxions to obtain the probabilities of survivorships of two or more lives, assuming the functions of life to be of the same continuous character. He investigates a problem to determine what would be the law of mortality between two lives A and B, so that, should it be known that they are both extinct, there would be an equal chance which of them died first, and it was found that such relation could only exist when the decrements for each life proceed in geometrical progression. The fluxional calculus is then applied to a number of complicated contingencies and survivorships, involving two, three, and more lives, which had been previously investigated by Mr. William Morgan† and Mr. Baily.‡ A scholium is added, in which the valuation of annuities payable more than once a year, and of assurances payable at certain fixed periods after death, are dealt with. This important paper has hardly met with the amount of attention it so fully deserves. The fact can, however, easily be explained, if we but bear in mind that the *Philosophical Transactions* are not accessible to everyone, and that the notation, as well as the mode of reasoning, are of a somewhat abstruse character.

A sequel to this paper, contained in a letter to Mr. Francis Baily, and read before the Royal Society on the 16th June, 1825, was destined to play a more important part. It treats on the nature of a function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. Mr. Gompertz having remarked that for certain intervals the law of mortality approaches nearly to a geometrical progression, considers the propriety of looking upon it as involving the actual law of nature. He establishes the law *à priori*, by supposing a person's resistance to death to decrease as his years increase, in such a manner that at the end of equally infinitely small intervals of time, he loses equally infinitely small proportions of his remaining power to oppose destruction. He further supposes that, among any given number of persons of equal vital powers, the probability of death is the same, but that it is inversely proportional to the vitality.

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* *Vide Philosophical Transactions* for 1825, p. 526.

† *Philosophical Transactions*, 1788, 1789, 1791, 1794, and 1799.

‡ *The Doctrine of Life Annuities and Assurances*, chap. viii.

It is to be regretted that in the analytical investigation which is given errors have crept into the text. As Mr. Gompertz was not able to carry out his intention of publishing a list of errata of these Tracts, notwithstanding the hope he held out in his letter published in the *Assurance Magazine* (vol. ix., p. 296), an exact transcript of the demonstration contained in Mr. Gompertz's own copy is herewith given, with a view to enable those that possess the work to correct the errors contained in the proof of the fundamental proposition.

"If L_x be the number of living at the age x, we shall have

$aL_x \times q^x\dot{x}$ for the fluxion of the number of death $= -(L_x)^{\cdot}$;

$$\therefore a\dot{x}q^x = -\frac{\dot{L}_x}{L_x};$$

$\dfrac{aq^x}{\text{log. of } \frac{1}{q}} = \text{hyp. log of } \dfrac{L_x}{d}$, d being a constant: therefore put $a \div$ com. log. of $\frac{1}{q} \times$ square of hyp. log. of $10 = c$, $cq^x =$ com. log of $\frac{L_x}{d}$; $\therefore$ putting $10^c = g$, $L_x = d.g^{q^x}$."

This law, which it will be seen is based upon a physiological principle of high probability, leads to conclusions which agree very satisfactorily with observed facts. To use the words of Professor De Morgan,* "Had the law been propounded in the days of Newton, *vitality* would have been made a thing of, like *attraction*."

In recent times, the formula has been extensively used for purposes of interpolation and adjustment of tables of mortality;† but nothing will more clearly exhibit its superiority to other laws, that have been presented by different authors, than actually placing these before the reader for comparison.

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* The *Athenæum*, July 22, 1865.

† *Tables of Mortality derived from the Experience of the Amicable Society*, by Thomas Galloway. *Life Contingency Tables*, by E. J. Farren. *Rates of Premium for Assurances of Military Officers in Bengal*, by Charles Jellicoe; *Assurance Mag.*, vol. i., p. 166. *Mortality prevailing in Eagle Insurance Company*, by Charles Jellicoe; *Ass. Mag.*, vol. iv., p. 199. *Mortality prevailing in Royal Insurance Company*, by Percy M. Dove.

Passing over D'Alembert's, D. Bernoulli's, and Duvillard's analytical investigations, which had for their chief object the determination of the mortality produced by the small-pox, we have the following formulæ; x being the age, and y the corresponding number of the living.

I. De Moivre's[*] :—

$$y=86-x.$$

II. Lambert's[†] :—

$$y=10000\left\{\frac{96-x}{x}\right\}^2-6176\left\{e^{-\frac{x}{13\cdot682}}-e^{-\frac{x}{2\cdot431}}\right\}.$$

III. Thomas Young's[‡] :—

$$y=368+10x-11\{156+20x-x^2\}^{\frac{3}{2}}+\frac{1}{2\cdot85+2\cdot05x^2+2\left(\frac{x}{10}\right)^6}$$
$$-5.5\left(\frac{x}{50}\right)^{10}+\frac{(5\cdot5)^2}{4000}\left(\frac{x}{50}\right)^{20}-5500\left(\frac{x}{100}\right)^{40}.$$

IV. Babbage's[||] :—

$$y=6199\cdot8-9\cdot29\frac{x}{1}-1\cdot5767\frac{x(x-1)}{2}.$$

V. Littrow's[§] :—

$$y=598\cdot1673-8\cdot4175x+\cdot2309x^2-\cdot0052x^3+\ \ldots.$$

VI. Moser's[¶] :—

$$y=1-ax^{\frac{1}{4}}+bx^{\frac{9}{4}}-cx^{\frac{17}{4}}-dx^{\frac{25}{4}}+ex^{\frac{33}{4}};$$

in the case of Brune's *Life Tables*, $a=\cdot2$, $b=\frac{\cdot7125}{10^5}$, $c=\frac{\cdot1570}{10^8}$, &c.

VII. Gompertz's original formula :—

$$y=d.g^{q^x}.$$

VIII. Makeham's[**] modification thereof :—

$$y=d.g^{q^x}s^x,$$

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[*] *Annuities on Lives*, by Abraham De Moivre. London, 1727.

[†] Gaeta and Fontana: *Dottrina degli azzardi.* Milano, 1776.

[‡] *Philosophical Transactions*, 1826, Part III.

[||] See Note by Mr. Peter Gray: *Assurance Magazine*, vol. vi., p. 185.

[§] Professor Littrow: *Ueber Lebensversicherungen und andere Versorgungsanstalten.* Vienna, 1832; p. 52.

[¶] Dr. Ludwig Moser: *Die Gesetze der Lebensdauer.* Berlin, 1839; p. 315.

[**] *Assurance Magazine*, vol. viii., p. 301; and vol. xii., p. 315.

It would carry the reader too far if much commentary were added to these formulæ. They not only compare unfavourably with Gompertz's law, on account of being purely empirical, but also on account of their being (with the exception of De Moivre's law) of so unwieldy a character as to be altogether unfit for practical use. Besides, these laws will only bear comparison with individual tables of mortality.*

Mr. Gompertz, in his contribution to the Royal Society's *Transactions* of 1825, examines the coincidence of his law with several tables of the best authority, such as Deparcieu's, the Northampton, Carlisle, and Swedish tables, obtaining in each case the necessary constants, by what he terms his "Vital Rule of Three."

In the second chapter, Mr. Gompertz points out how annuities on several lives may be computed by the use of the tables called by him "accommodated ratios," wherein he assumes the number of persons living at equal intervals of successive ages to be in simple geometrical progression, the periods being taken sufficiently short to permit this assumption to be a near approximation to truth.

This important paper did certainly not meet at first with the attention it so fully merits. To Professor De Morgan no little credit is due, not only for popularizing the subject, but for finding out some interesting properties of the law.†

Messrs. Woolhouse's, Sprague's, and Makeham's labours should also here be mentioned with much praise. But to Mr. Gompertz himself we are indebted for the greatest improvements in and extension of his law. He was, however, not enabled to resume the investigation of this subject till about the year 1860. During this long interval of 35 years he was so much engaged in the management of life assurance concerns that he found little leisure for private study.

When the Guardian Assurance Office was established, Mr. Gompertz applied for the situation, but Mr. Griffith Davies was the successful candidate. So far was Mr. Gompertz from feeling piqued towards his competitor, who had been in his way on several previous occasions, that he was all the more obliging. Eventually he introduced him at the Royal Society, and soon after carried his election as a Fellow.

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* Mr. T. R. Edmonds, in the *Philosophical Magazine* for January, 1866, professes to give a new expression for the law of human mortality, derived from analogy with the law of density of saturated steam. The question of the accuracy or novelty of this law does not concern us. We regret, however, that when speaking of his old formula, he omits making any mention of, much less doing justice to the subject of our Memoir. It is a noteworthy circumstance, that Mr. Edmonds, in the course of the paper, no longer insists upon the fact that life is made up of three periods—infancy, florescence, and senescence; but contends for two periods of life only—childhood and manhood.

† *Vide Penny Cyclopædia*, art. "Mortality"; *Philosophical Magazine*, Nov., 1839; *Assurance Magazine*, vol. viii., p. 181.

In the year 1823, when he suffered under the severe visitation of the death of his only son—a boy, then, nine years of age—his friends felt that he could only gain relief of mind by altogether devoting himself to a profession which seemed to be most congenial to his feelings. Chiefly through the endeavours of his friends, Sir Moses Montefiore, Messrs. N. M. Rothschild, and Samuel Gurney, he was appointed actuary to the Alliance British and Foreign Assurance Company, which was founded at that time.

Having had only one month allowed him to prepare all the rates and regulations of the Company, he without hesitation adopted the Carlisle table of mortality, which he found satisfactorily to agree with the Equitable Experience. The Office under his guidance made most satisfactory progress. One measure the directors of his Company were anxious to promote at the time—viz., to compete with Government as to the grant of annuities. This he opposed strenuously. The terms on which the Government were granting these annuities were based on the Northampton tables, which, though safe enough for assurances, could not but entail ruinous consequences with regard to annuities. Eventually he even convinced the able Government Actuary, Mr. John Finlaison, of the heavy loss that was caused by using these tables; and in the year 1829 those rates were superseded by the higher ones, based on the mortality experienced by Government Annuitants.

Mr. Gompertz's views on this subject are stated very clearly before a Parliamentary Committee that sat in the year 1827.*

In a sketch of Mr. Gompertz's life in the *Athenæum* of the 22nd July last, reference is made to another actuary of eminence, the late William Frend, of the Rock Office, who, though on friendly terms with Mr. Gompertz, was his opponent in more than one respect. In matters of life assurance the one may be considered the champion of the Carlisle, the other, in conjunction with Mr. Morgan, of the Northampton table of mortality. Mr. Gompertz was as staunch a Newtonian as Mr. Frend was an anti-Newtonian. Both may be called the last of their class. Again, with regard to imaginary quantities, we have seen Mr. Gompertz's labours in that field. Mr. Frend also directed his attention to that subject, but with a view to the rejection of imaginary as well as of negative quantities.†

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* *Vide* Minutes of Evidence before the Select Parliamentary Committee on the laws respecting Friendly Societies, 1827.

† *Principles of Algebra.* London, 1796.

The ability with which Mr. Gompertz conducted Office valuations, as well as other actuarial calculations, may be seen from the fact that, at the public sales of reversions, &c., his opinion ruled, more or less, the market. The mode in which he treated questions for the sale of life annuities and contingent reversionary property is interesting, but a description thereof would carry us too far.

As may be supposed, he was frequently referred to by Friendly Societies and kindred institutions for advice. For the Army Medical Board, by which he was consulted, he made elaborate computations, and revised the rates they charged in connexion with the Fund for Widows and Orphans.

We must also mention the prominent part he took in promoting the publication of the Experience Table of the 17 Assurance Companies, which appeared in the year 1843.

The transfers of life interests and reversions were becoming so frequent that he advised the establishment of a Company, whose exclusive business it should be to deal in such property. Accordingly, the National Reversionary Investment Company was founded, and he continued its actuary from its establishment to the time that ill health made it necessary for him to retire. In the year 1847 he also relinquished the actuaryship to the "Alliance."

He continued, however, sedulously to attend the meetings of the different learned Societies to which he belonged, and otherwise to promote their interest. Amongst these Societies we have to mention the Institute of Actuaries, of which he was an honorary member; the Statistical Society, in the establishment of which he took part; the Royal Literary Fund, of which he was steward on several occasions; the Society for the Diffusion of Useful Knowledge, on the Committee of which he served from the year 1828; the new London Mathematical Society, for which he was actually preparing a paper when attacked by his last illness.

During the ten years that immediately followed his retirement from active duties, we do not hear so much of his original investigations and labours; in fact he suffered so much from ill health, that his chief occupation consisted in diverting himself with mechanical contrivances, in which even as a boy he had delighted.

At length his complaint, the gout, became so serious, that he was quite unable to move from his room; and during the last six years of his life he did not so much as leave his bed-room floor. He bore this confinement with wonderful patience; and the more his bodily strength failed, the greater became the activity of his mind.

In the year 1860 he presented to the International Statistical Congress which met at London, a paper on one uniform law of mortality from birth to extreme old age; and the ideas contained therein were further elaborated in the first supplement to his papers of 1820 and 1825, that were presented to the Royal Society.

Besides many other valuable investigations, he gives a more complete formula for the law of mortality, involving several terms, which may be reduced to the form we are already acquainted with, viz., $y=dg^{q^x}$, and yet express the mortality from the commencement of life to extreme old age, if d g q, instead of being constant, are supposed to vary slowly.

Gompertz's improved law is presented under the following two forms—

$$y=\text{constant}.\,\mathrm{A}^{e^x}\mathrm{B}^{e_1^x.x}\mathrm{C}^{q^x}\mathrm{D}^{\mathrm{P}_x},\ \text{where } \mathrm{P}_x=\theta\omega^{\pi^x.\overline{x-u}};$$

$$\lambda y=\mathrm{C}\beta^x+\underset{11}{k\epsilon^x}+k\epsilon^x-\lambda^{-1}(e^x\lambda q_0.\overline{x-h})+\mu\nu^x.$$

where from birth to extreme old age all the quantities but x and y are constant.

$\underset{11}{k\epsilon^x}$ commences its significance at birth, and is particularly interesting, because in the Carlisle table it shows a surprising agreement with the stated mortality of children from birth till the age of 1 year.

$k\epsilon^x$ commences significance at birth, but gradually sinks; and at age 20 and after sinks into entire insignificance.

$\mathrm{C}\beta^x$ and $\lambda^{-1}(e^x\lambda q_0.\overline{x-h})$ are of significance from birth to extreme old age.

$\mu\nu^x$ is insignificant till x equals about 80, and then slowly increases during the remainder of life.

These constants are obtained by a peculiar process of differencing, an extension of the so-called "vital rule of three."

Now some people think Gompertz's law untenable, because the mortality of man, or at all events of certain classes, does not increase by a steady progression, but exhibits certain climacteric departures at certain periods of life. That such is the case cannot be doubted. Similar climacteric departures are exhibited even at infancy. The mortality at birth is much greater than that at a few years later, which again increases before the period of puberty.

Now, we have just seen how Gompertz's extended law applies to those cases, and the author ventures to assert, that by applying

Gompertz's peculiar process of differencing at certain stages of life, the law will be found to adapt itself to the exceptional mortality at those ages.

In the Supplement under consideration methods are also pointed out by which the most intricate questions of life contingencies, involving any number of lives, may be solved without difficulty.

An instructive consideration of questions of special risk, and what he terms "special influenced contingencies," which have an important bearing upon survivorships, is also given. In conclusion, he shows how his theoretical law of mortality not only applies to infancy, but holds also for sickness. He finds that the values obtained from his "vital rule of three" compare most satisfactorily with the tables deduced by Messrs. Ansell and Neison from the experience of Friendly Societies.

A second supplement, placed before the Royal Society the 12th May, 1864, contains some valuable suggestions on the Commutation Tables of Barrett; and professes to give a method by which the existing competition amongst Assurance Companies, which the author considers injurious to their interests, may be checked.

Though his mind had not been affected, his bodily state of health began at last so to give way, that he could not complete this last paper to his satisfaction. In April this year he had a stroke of paralysis, and then indeed his days were numbered. He lingered on till the 14th July, when a calm death released him from further sufferings. He was buried a few days afterwards, with every mark of respect, in the Jewish burial ground near Victoria Park.

To do full justice to Mr. Gompertz's character is no easy task. We have adverted to one or two incidents in his life, wherein he showed true nobleness of disposition and kindness of heart. But these qualities were not the results of the occasional promptings of his feelings, they characterized all his actions.

It is not too much to say, that he was never known to decline giving his assistance in the cause of charity, and this aid he offered to an extent which far exceeded his means. But true charity is not limited to the giving of mere alms, it consists chiefly in aiding by the powers of the intellect. We accordingly find him a prominent member of some of the leading Jewish charities. He also worked out a plan, which has eventually been acted upon, by which the dispensation of charity is placed in the hands of a Board of Guardians, whose duty it is to investigate the claims and keep a register of all applicants for relief, to assist them in distress, and to procure them employment.

What a kind husband, what a loving father, Mr. Gompertz was, only the immediate members of his family know: those to whom he extended his friendship were at a loss which more to admire, his social qualities or mental attainments.

May science ever find votaries as worthy and ardent as the late Benjamin Gompertz!

On the Limitation of Risks; being an Essay towards the Determination of the Maximum Amount of Risk to be retained by a Life Insurance Company on a Single Contingency. By THOMAS BOND SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 29th January, 1866.]

IT is my object in the present paper to discuss the principles which should guide the actuary in fixing the maximum amount of risk to be retained by a Life Insurance Company upon a single contingency.

Those principles are by no means limited to life insurance, but extend with suitable modifications to other kinds of insurance, as fire, marine, hailstorm, health, and accidental death insurance. They are also applicable to the risks of mercantile transactions, and in particular to the risks incurred by a bank in discounting its customers' bills. There can be little doubt that some recent failures of banks have been caused principally by the neglect of these principles; in other words, the manager has failed to put a proper limit on the risks undertaken, *i.e.*, has trusted individuals to an imprudent extent, and the loss of heavy sums consequent on their failure has proved disastrous to the bank.

The risks attending life insurance however differ in several important respects from the other classes of risks above mentioned. In particular, there cannot be a "partial loss," as there may be in those other businesses. An ordinary life policy differs also from other kinds of insurance in the following respects—that the contract entered into by the Office issuing the policy, does not run out in a fixed time, and cannot be terminated at the option of the Office at stated periods like a fire insurance contract. Again, the risk increases yearly during the continuance of the policy; and lastly, it is certain that in every case a claim will sooner or later arise. For these reasons it will be convenient to limit our attention on this occasion to the risks of life insurance only.

As a first step in the discussion, let us seek an answer to the inquiry:—"Why is it necessary for a Life Office to put any limit at all upon the risks it undertakes? Why should not an Office, for example, issue a policy for £100,000, and run the risk for the whole amount?" At first sight it may seem sufficient to answer, that no Office should run the risk of being called on to pay a sum which it would be impossible or inconvenient for it to provide. But very little consideration is required to show that this is not a satisfactory or a complete reply. For every Insurance Company does, and must, from the nature of its business, run the risk of being called on to pay sums which it would be impossible for it to realize. If all the policies issued by any Company, or even a large proportion of them, were to become claims simultaneously, the Company would assuredly be ruined. But this chance of ruinous loss is one which the Company may properly incur. To take an example, it cannot be denied that a Company with a paid-up capital of £20,000 would be quite justified in becoming liable for £100,000 under 100 policies of £1,000 each, or under 1,000 policies of £100 each; but it would not be justified in becoming liable for the same sum under 10 policies of £10,000 each, and certainly not under 2 policies of £50,000, or a single one of £100,000.

What is it, then, that constitutes the difference between these various cases? We will suppose for the present that all the insurances are for a single year only. Then the most obvious difference is that the more numerous the insurances are which constitute the total risk, £100,000, the less is the chance of the Company being called on to pay the whole. If the rate of mortality be taken at 1 per cent. per annum,* or the chance of dying

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* The mortality being such that out of 1,000 persons alive at the beginning of a year 10 die during the year, it appears correct to say, either that the "annual mortality is one per cent.," or that "the rate of mortality is one per cent. per annum," or that "the annual rate of mortality is ·01." In the latter forms of expression, the phrase "rate of mortality" is used in a sense strictly analogous to the common one "rate of interest." Thus we say that "the rate of interest is five per cent. per annum," if £1,000 has by the operation of interest become £1,050 at the end of a year.

It must be borne in mind by the student, in reading the Reports of the Registrar-General and similar works, that the term "rate of mortality" is used by Dr. Farr—most injudiciously and improperly in my opinion—to denote a different quantity:—that, namely, which has been called by Mr. Gompertz the "intensity of mortality," and by Mr. Woolhouse (very happily) the "force of mortality." It is not easy to give in in a year $= \cdot 01$, then the probability of the Company being called on to pay £100,000 under a single policy for that amount is ·01. If there are 10 policies of £10,000 each, on different lives, the probability of their all falling in will be $\cdot 01^{10}$; and if there are 100 policies of £1,000 each, the probability of the Office being called on for the whole £100,000 is $\cdot 01^{100}$, which is an excessively small quantity with 199 ciphers following the decimal point.

This, however, is a very incomplete comparison; and we shall get a better idea of the difference between the various cases, if we calculate in each the chance that the loss will be not less than a given amount—say £10,000. Thus, when there are 10 policies of £10,000 each, the chance that there will be no claim is $\cdot 99^{10}$, which is equal to ·90438; and therefore the probability of at least one loss occurring is ·09562, which is the probability that the Office will be called on to pay £10,000, or more.

Next, when there are 100 policies of £1,000 each, denoting by P_x the probability that x claims will arise, we have

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popular language a strict definition of the "force of mortality;" but it is approximately equal to the ratio which the number dying in a year bears to the number living in the middle of the year.

To enable the reader to understand the distinction more readily, I quote the remarks of Dr. Farr, given in the Blue Book, on the Sanitary State of the Army in India (p. 37):—"In the reports upon the Indian Fund, the probability of dying is incorrectly called the *rate of mortality*, so as to mislead the unwary reader. Thus, if on an average out of 100 men living at the beginning of a year there are 10 deaths in the year following, the probability of dying is expressed by the fraction $\frac{10}{100}$, which is incorrectly called in the reports 'the rate of mortality'; but the rate of mortality is $\frac{10}{95}$, for the numbers living at the end of the year are 90, and the years of life are 90 + 10 half years, which it may be assumed are lived by the 10 who died in the course of the year." This appears to me much as if we should say, "If £100 has by the operation of interest become £105 at the end of a year, the rate of interest is not $\frac{5}{100}$, but $\frac{5}{102\frac{1}{2}}$." In my opinion Dr. Farr is wrong, and the authors he criticizes are right, in the meaning they attach to the phrase, "rate of mortality."

I have not been able to ascertain that any writer, with the exception of Mr. Edmonds, agrees with Dr. Farr in his views on this point; and I conclude that the attempt to attach a new meaning to the familiar phrase, "rate of mortality," is an innovation much to be deplored, and in every way to be discountenanced. By this and similar innovations—as his definition of an "Annuity," and his alteration of the "N Column"—Dr. Farr has greatly diminished the usefulness of his writings.

If L_x denote the number living at any age x, integral or fractional, then the annual "rate of mortality" at that age x will be $\frac{L_x - L_{x+1}}{L_x}$; and the "force of mortality" at the same age, $\frac{1}{L_x} \cdot \frac{dL_x}{d_x}$. If the mortality follow Gompertz's law so that $L_x = dg^{q^x}$, then the annual rate of mortality at the age x will be found to be $1 - g^{(q-1)q^x}$; and the force of mortality, $\log_e g . \log_e q . q^x$.

P_0[*] $=$ ·3660323
$P_1 =$ ·3697296
$P_2 =$ ·1848648
$P_3 =$ ·0609992
$P_4 =$ ·0149417
$P_5 =$ ·0028978
$P_6 =$ ·0004634
$P_7 =$ ·0000628
$P_8 =$ ·0000074
$P_9 =$ ·0000007

The sum of these probabilities being ·9999997

Here we see that the chance that there will be no claim is rather more than one in three; the chance of a single claim arising is also rather more than one in three; and the chance of more than one claim arising is less than one in three. We see also that the probability of the losses amounting to £10,000 or more, is about ·0000003.

This example shows us how rapidly the probability of a dangerous loss diminishes when the number of risks is increased. On the other hand, it must be noticed that the probability that no loss will arise, diminishes rapidly as the number of risks increases. In the case of a single policy, the chance of there being no loss is ·99. When there are ten policies, the probability that there will be no claim is $\cdot 99^{10}$ or ·90438. When there are 100 and 1,000 policies, the same probabilities are ·3660323 and ·000043171 respectively.

Thus then we see that, as the number of policies is increased—the sum assured remaining the same—the probability of a large loss occurring is diminished, and also the probability of a very small loss occurring. The effect of increasing the number of

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[*] These values have been calculated as follows:—$P_0 = \cdot 99^{100}$, and is easily found by logarithms. Then

$$P_1 = \frac{100}{1} \times \cdot 99^{99} \times \cdot 01 = \frac{100}{99} P_0 = \frac{100}{100-1} P_0$$

$$= \frac{P_0}{1-\frac{1}{100}} = P_0 \left\{ 1 + \frac{1}{100} + \frac{1}{(100)^2} + \ldots\ldots \right\}$$

$$= P_0 + \cdot 01 P_0 + \cdot 0001 P_0 + \&c.$$

$$P_2 = \frac{100 \times 99}{1 \times 2} \times \cdot 99^{98} \times \cdot 01^2 = \frac{1}{2} \times \cdot 99^{99} = \frac{P_1}{2}$$

$$P_3 = \frac{100 \times 99 \times 98}{1 \times 2 \times 3} \times \cdot 99^{97} \times \cdot 01^3 = \frac{98}{3} \times \frac{1}{99} P_2$$

$$= \frac{1}{3} \times \frac{99-1}{99} \times P_2 = \frac{1}{3}\left(1 - \frac{1}{99}\right) P_2$$

$$= \frac{1}{3}\{1 - \cdot 01 - \cdot 0001 - \&c. \ldots\ldots\} P_2$$

$$P_4 = \frac{97}{4} \times \frac{1}{99} \times P_3$$

&c. &c.

policies, is to diminish the probability of a large departure from the average in either direction; and the larger the number of lives under observation, the less is the chance of an accidental fluctuation in the number of deaths.

It is not my intention on the present occasion to pursue our subject further in this direction, and to seek by the aid of the higher mathematics, a measure of the rate at which the probability of a fluctuation decreases when the number of cases is increased, as is done by De Morgan in his treatises on the *Theory of Probabilities.* I have to confess, indeed, that I do not consider the results of those investigations of much practical importance in their application to the theory of life contingencies. In fact, we must always be careful in reasoning on that subject that we do not apply our mathematical conclusions too unreservedly; we must always bear carefully in mind the suppositions and limitations under which they have been obtained. For example, if we trace from year to year the number of deaths in the population at large, we shall often find greater fluctuations than the theory of probabilities would lead us to expect—the fact being that our fundamental supposition of the probability of death being the same in different years at the same age, is rendered untrue by the occurrence of unhealthy years, or years of scarcity and distress. And as the number of persons under observation in a Life Insurance Company increases and becomes very large, we may expect that while accidental fluctuations arising from the paucity of members will become of less consequence; on the other hand, fluctuations may be anticipated from the same causes as produce them in the number of deaths among the population at large.

The practical conclusion at which I have arrived then is that in all Insurance Companies, whether small or large, fluctuations in the mortality must be expected; and a proper provision should be made for them in the premiums charged.

We have seen that the effect of increasing the number of policies, the total sum assured remaining the same, is to lessen the probability of a very large loss or of a very small loss occurring; and we shall find that the "expectation of loss" remains unchanged, being in all the cases we have considered above, equal to the premium for the risk, £1,000. It will be worth while to elucidate this point further.

1. When there is a single policy for £100,000 the Office can only lose the whole or nothing. The chance of losing the whole is ·01; and the expectation of loss $=$ ·01 × £100,000 $=$ £1,000.

2. When there are 10 policies for £10,000 each, the Office may lose either

£100,000, £90,000, £80,000, ... £10,000, or nothing;

and the probabilities of these several losses are—

$$\cdot 01^{10},\ 10\times\cdot 01^{9}\times\cdot 99,\ 45\times\cdot 01^{8}\times\cdot 99^{2},\ \ldots\ldots\ \cdot 99^{10},$$

being the terms of the expansion of $(\cdot 01+\cdot 99)^{10}$.

3. When there are 100 policies of £1,000 each, the Office may lose either

£100,000, £99,000, £98,000, ... £2,000, £1,000, or nothing;

and the probabilities of these several losses are the terms of the expansion of $(\cdot 01+\cdot 99)^{100}$.

We have now to prove that in these latter cases, as also when there are 1,000 policies of £100 each, the expectation of loss is still £1,000. In order to do this, let us suppose that a sum S is assured under any number n of policies; and that the chance of any one becoming a claim is x, then the possible losses are—

$$S,\ \frac{n-1}{n}\cdot S,\ \frac{n-2}{n}\cdot S\ \ldots\ldots\ \frac{2S}{n},\ \frac{S}{n},\ 0;$$

and the probabilities of these several losses are

$$x^{n},\ nx^{n-1}(1-x),\ \frac{n.\overline{n-1}}{1.2}\,x^{n-2}(1-x)^{2},\ \ldots\ nx(1-x)^{n-1},\ (1-x)^{n}.$$

The total expectation of loss is therefore

$$S\times x^{n}+\frac{n-1}{n}S\times nx^{n-1}(1-x)+\frac{n-2}{n}S\times\frac{n.(n-1)}{1\cdot 2}x^{n-2}(1-x)^{2}$$
$$+\ \ldots\ldots\ +\frac{S}{n}.nx(1-x)^{n-1}$$
$$=Sx\{x^{n-1}+(n-1)x^{n-2}(1-x)+\frac{(n-1)(n-2)}{1\cdot 2}x^{n-3}(1-x)^{2}$$
$$+\ \ldots\ldots\ +(1-x)^{n-1}\}$$
$$=Sx\{x+(1-x)\}^{n-1}=Sx\times 1^{n-1}=Sx.$$

Thus we see that the total expectation of loss is independent of n, the number of policies; and in the case we have been considering, where $S=100{,}000$ and $x=\cdot 01$, the expectation of loss is £1,000, whatever the number of policies.

Before proceeding further we pause to point out that this "expectation of loss" is a different thing from the "most probable loss." In fact, in many cases it is impossible for the Office to lose the exact amount of the "expectation of loss." To take an example, if £100,000 is insured under 625 policies of £160 each, it is not possible for the Office to lose exactly £1,000. In order to find the amount of the most probable loss, we will resume our former supposition of a sum s, insured under n policies of equal amounts, then the probability of a loss of $\frac{r\mathrm{S}}{n}$ occurring is

$$\frac{n.\overline{n-1}.\overline{n-2}\ \ldots\ \overline{r+1}}{1.2\ \ldots\ \overline{n-r}}\,x^r(1-x)^{n-r},$$

and the probability of a loss of $\frac{r+1}{n}\mathrm{S}$ is

$$\frac{n.\overline{n-1}.\overline{n-2}\ \ldots\ldots\ \overline{r+2}}{1.2\ \ldots\ldots\ \overline{n-r-1}}\cdot x^{r+1}(1-x)^{n-r-1};$$

and the latter probability is greater or less than the former

$$\begin{array}{lr}
\text{according as} & \frac{n-r}{r+1}\cdot\frac{x}{1-x} >< 1, \\
\text{according as} & (n-r)x >< (r+1)(1-x), \\
\text{according as} & nx-rx >< r+1-rx-x, \\
\text{according as} & nx+x-1 >< r, \\
\text{according as} & r <> nx+x-1.
\end{array}$$

Now, as r increases from 0 to n, the fraction $\frac{n-r}{r+1}\cdot\frac{x}{1-x}$, decreases from $\frac{nx}{1-x}$ to 0; and if $\frac{nx}{1-x}>1$, then the fraction will be first >1 and afterwards <1. Hence, as r increases, if $\frac{nx}{1-x}>1$, the probability of r claims arising will first increase and then diminish, and will be greatest when r is the integer next greater than $(n+1)\,x-1$; or is the integral part of $(n+1)\,x$. If $(n+1)\,x$ happen to be an integer, then we conclude that $(n+1)x-1$ and $(n+1)\,x$ claims are equally probable, and more probable than any other number of claims.

Returning now to the supposition of 625 policies for £160 each, we have $x=625$, $x=\cdot01$, or $(n+1)\,x=6\cdot26$, so that the most probable number of claims is 6, giving a loss of £960. Here, as was to be expected, the "most probable loss" does not differ much from the "expectation of loss," which we have already seen to be £1,000. In this instance $\frac{nx}{1-x}=625\times\frac{\cdot01}{\cdot99}=\frac{625}{99}$ and is therefore greater than unity.

As an illustration of this case where $(n+1)\,x$ is an integer, let us take $n+1=700$, or $n=699$, so that we have 699 policies of £143·0615 each. Then, since $(n+1)x=7$, we learn from what precedes that 6 and 7 claims are equally probable, and more probable than any other number. In those cases, the losses would be £858·37 and £1,001·43 respectively.

We conclude also, comparing the results of the two last paragraphs, that if the number of policies is between 625 and 699, then the most probable number of claims is 6; and it is easy to see that 6 is the most probable number of claims if the number of the policies be anything from 600 to 698 inclusive.

Again, when there are 1,000 policies of £100 each, we have $(n+1)\,x=10·01$, or the most probable number of claims is 10, and the most probable loss £1,000, the same as the expectation of loss. In general, the expectation of loss is $\mathrm{S}x$ or $nx \times \dfrac{\mathrm{S}}{n}$, and the most probable loss is $\left\{\text{the integral part of } (n+1)\,x\right\} \times \dfrac{\mathrm{S}}{n}$. If, then, nx is an integer, it is of necessity the integral part of $(n+1)\,x$; and the expectation of loss is equal to the most probable loss.

It will be noticed that the "expectation of loss" is analagous to the "expectation of life at a given age," called by some writers the "mean duration of life." The "most probable loss," is analagous to the "most probable duration of life"—that term of years which is more likely than any other to be the duration of life. This is not the same as the term of years (called by French writers "*la vie probable*") which it is as likely as not that a given person will survive. (See Gray's *Tables and Formulæ*, §§ 162, 163, where the difference between these various quantities is very clearly explained.) To take an example: By the Carlisle Table a person of the age 30, has an "expectation of life" equal to 34·34 years. But since, according to the Carlisle Table, more persons over the age of 4 die at the age of 74 than at any other age, the "most probable duration of life" is 44·5 years. Again; since of the persons living at the age 30, one half attain the age of 66 but not 67, it follows that the time which a person 30 years of age has an even chance of surviving, is a little over 36 years. We have not considered in what precedes the quantity which is analogous to this one; it is "the loss which it is an even chance that the claims of the year will not fall short of."

We return now to the question, "Why is an Office with a capital of £20,000, justified in becoming liable for £100,000 under 100 policies of £1,000 each, but not under 10 policies of £10,000 each?" The preceding investigations enable us to say that the chance of the Office sustaining a dangerous or ruinous loss, is very much less in the former case than in the latter; and is, in fact, so small in the former case, that it may be disregarded.

Can we then conclude generally that a Company is justified in running the risk of having to pay a very large sum on the happening of any extremely improbable contingency? For example, would a Company be justified in granting a policy for £100,000 to be payable only in case a young man, now 20 years of age, should die on his 90th birthday, or in case all of 20 specified persons should die within a year in a specified order? . . . . (A).

It will no doubt be answered by some persons that a Company would not be justified in undertaking insurances of the nature of those mentioned in the last paragraph, because such insurances, if ever made, would be rare, and could never be sufficiently numerous to enable the Company to obtain an average result from them. To this argument I cannot myself allow any weight. I can find no sufficient reason for thinking that it is necessary for an Insurance Company to have a sufficient number of each class of risks, to form a separate average. On this point I entirely agree with the remarks of the late Dr. Young, given in the *Assurance Magazine*, vol. vi., p. 289. "There is a common prejudice that it is disadvantageous for an Office to take a single risk of any particular description; and it is sometimes said that if the adventure should happen to be unsuccessful, there would be no possible compensation from others of the *same* kind. There is however just the same chance that it would be compensated by others of a *different* kind."

I do not propose to discuss this point at the present time, but will content myself with an illustration. In the course of conversation with a merchant, I learned that he did not insure his house or furniture against fire; and on my inquiring the reason, he said, "If my house were burnt I might lose £1,500 or £2,000; but I run as great or greater risks in the ordinary course of my business. I may any day lose £2,000 by the failure of a single customer; and if I run that risk, I may fairly run the risk of loss by fire too." Subsequent consideration of this argument has led me to question its soundness, on the ground of the insufficiency of the consideration for the risk. While admitting that risks of wholly different kinds may fairly be set off the one against the other, I would yet ask, "Is it worth while for a merchant to run the risk of losing, say £2,000, by fire, his consideration for the risk being the premium of insurance which he saves?" The amount he saved by running his own risk may be 4*s.* 6*d.* or 5*s.* per £100 per annum; but the consideration for the risk of a loss in the ordinary course of trade must have been far larger than this. The ordinary profits of trade cannot, I should suppose, be taken at less than 10 per cent. per annum—subject to deduction for bad debts. I have no means of estimating the probability of bad debts arising, but should suppose it cannot ordinarily exceed 2 or 3 per cent. per annum. Here then there would be a profit of 8 or 7 per cent. per annum, which may be considered as the consideration for incurring the risk of a bad debt; and by the side of this, the consideration of ¼ per cent. for running the risk of loss by fire, seems quite contemptible. On *this* ground, then, but not on account of the diverse characters of the risks, I conclude that it is not worth while for a merchant to be his own insurer against fire.

The conclusion I arrive at, then, on this point, is that risks of various kinds may be fairly classed together, when the amounts of the several risks are approximately equal, and also the several chances of loss. But when there is a great disparity in the chances of loss and in the premiums paid, then the risks cannot properly be classed together. For this reason, therefore, an Insurance Company cannot properly class together with life insurance risks those of fire insurance or accidental death—the premium for the first scarcely ever being less than £1 per cent. per annum; and in the second, commonly 1*s.* 6*d.* per cent. per annum; while in the last, a common premium is 3*d.* per £1,000 or ·00125 per cent. If then a Company transacted business of these three kinds, there should be enough risks of each class to form a separate average.

But on the other hand there seems to be no sufficient reason for the directors of a Life Office to decline to grant an insurance on the life of a negro, mulatto, or native Indian, on the ground that it is improbable there will be many such insurances effected.

We are now in a position to see that the question (A) must be answered by a reference to the consideration received for the risk.

In the case of £100,000 insured on the lives of 100 persons, if the premium for the risk is 1 per cent., or £1,000 in all, the premiums received will not generally be less than £1,250. But the net premium for an insurance of £100,000 on either of the very improbable contingencies mentioned in (A), would be an excessively small quantity; and the whole profit on the transaction so small, as to render it quite undesirable to entertain such a proposal and incur so large a risk. If, however, circumstances permitted of the same profit—say £250—being made on the single £100,000 policy as on the 100 other policies, it might become a question whether it would not be worth while for the Company to undertake the risk.

We may perhaps state more broadly the principle here suggested, and say that when the profit on a transaction is unusually large, an Office may with propriety take the risk of a larger amount than in ordinary cases. Thus the premiums received upon "issue insurances" yield a far larger margin of profit than the premiums in ordinary life insurances; and a Company might perhaps, on that ground, properly take a larger risk in those cases than it does in life insurances.

As a rule, then, we may say that (1.) an Office will not run the risk of £100,000 on a single life, because the probability of loss is too great to be prudently incurred.

(2.) That there is no objection to the Office becoming liable for £100,000 upon 100 or 1,000 different lives, because the probability of a dangerous loss occurring is small, and the profit on the transaction is adequate to the risk.

(3.) That the Office would decline to grant a policy for £100,000 on an extremely improbable contingency, although the chance of loss is very small, because the profit on the transaction must generally be too small to be worthy of regard.

The previous principles enable us to answer a question which arises sometimes when a Company issues its policy for a large amount, but retains only a portion of the risk, reinsuring the remainder. In this case, can the Office prudently issue its acceptance for the whole amount before it has arranged to be relieved of the excess of risk? We must say, certainly not! For the consideration the Company receives for running the whole risk for a time, is so small as to be unworthy of regard.

Having now satisfactorily ascertained the reasons for which it is necessary to place a limit on the amount of the risks undertaken by an Office, we proceed to consider how the particular limit suitable for any Office is to be ascertained.

For this purpose we will suppose that a policy of the maximum amount has become a claim, and will consider out of what fund the claim is to be paid. For the present, we will, as before, suppose all the policies to be effected for a single year, in which case the claims of any year are to be paid out of the premiums received in the year. Let us suppose also that the rate of mortality is one per cent. per annum, and examine how matters would stand if the business of the Office consisted of a few large policies—say of £10,000 each, and a large number of policies of £100 each. Then, if the premiums are loaded 25 per cent., the premium on each £10,000 policy will be £125; and on each £100 policy, £1·25; and if a policy for £10,000 becomes a claim, it will require the premiums of 80 such policies to satisfy the claim. If there are not so many £10,000 policies in force, the deficiency will have to be made good out of the premiums on the £100 policies; for example, if there are only 40 policies of £10,000, the premiums on these amount to £5,000, and it will require the premiums on 4,000 of the £100 policies to make up the other £5,000. But the premiums on the £100 policies are required to provide for the claims among those policies, and cannot therefore (except as regards the margin thereof) be considered available for payment of the £10,000 policy.

We thus arrive at the principle that the policies of maximum amount in an Office must be looked upon as forming a class by themselves; so that the claims arising among those policies are to be paid out of the premiums received upon them. It follows at once that the position of an Office cannot be considered satisfactory unless the policies of maximum amount are at least so numerous that the premiums received on them in the course of a year suffice to pay a claim, if one should arise; and the average premium being about 3 per cent., the very least number of maximum policies which an Office must have before its position can be declared satisfactory is 33. And unless the number of such policies is much greater than this, the Office will be liable to inconvenient fluctuation in the amounts of its annual claims. Supposing, first, that all the policies are effected for a single year only; then, if no claim occurs in the year, the whole of the premiums on the 33 policies will be profit; while if there should be a claim, it would absorb all the premiums, and there would be no profit on the year's transactions. This inconvenient fluctuation in the amount of the profits is to a very great extent remedied when a valuation for the distribution of profits is made only once in several years; for then it may be anticipated that the losses of one year will be compensated by the profits of others. Thus, if there are 33 policies of maximum amount under observation for three years before a valuation is made, this is, to a certain extent, the same as if there were 99 such policies under observation for a single year. On the whole, we may say that when an Office makes a valuation and distribution of profits only once in three, five, or seven years; or when it makes an annual valuation, but distributes not the profit of the past year, but an amount obtained from the average of several years; then it may be considered as practically secure from inconvenient losses or fluctuation of profits, if it has about 33 policies of maximum amount in force.

But no Company can hope, when it adopts a particular maximum, to obtain at once so many policies of maximum amount. What, then, is the smallest number with which it should start? It is well known that the mortality among recently selected lives is very light, and that it is five or six years before the effect of selection wears off; we may then perhaps allow an Office that period for obtaining the above mentioned number (33) of maximum risks. Thus we finally conclude that an Office ought yearly to obtain *at least* six or seven policies of maximum amount; and if the Office does not obtain that number, it must be inferred that its maximum has been fixed too high and should be lowered.

When the business of an Office is very extensive, the previous conclusions will require to be modified. Resuming the supposition of a mortality of 1 per cent., and supposing that the premium on a £100 policy is 1·25, we may say in general terms that the £1 is required to pay the claims, and the ·25 is available for expenses and contingencies; or we may consider that, after payment of expenses, the balance of the loading (·25) is available for payment of the £10,000 claims. If, then, the business is so extensive that the margins of the premiums, after payment of expenses, are equal to say three or four times the maximum amount of risk, we see that a claim of maximum amount might be paid out of the margins of the premiums on the smaller policies, without any greater inconvenience than a temporary diminution of the profits of the Office; and we thus arrive at the conclusion, that when the business of an Office is so extensive that the annual anticipated profit, will pay without inconvenience a maximum claim, then it is not essential that the maximum policies should be numerous. A Company of *very* large business might even prudently issue a single policy much larger than any other of its existing policies.

We have supposed above that all the policies were effected for a single year; but our conclusions are equally applicable to the ordinary insurances of a Life Office, if we substitute for the annual margins of the premiums, the annual anticipated profit, which may be stated in a general way to consist of the margins of the premiums, and of the excess of the interest realized over the rate of interest—3, 3½, or 4 per cent.—assumed in the calculations of the Office. But it must be borne in mind that if, in making a distribution of profits, the margin of the premiums has to any extent been encroached upon, it is to that extent non-existent.

The conclusion at which we have arrived above, viz., that the largest policies in an Office are generally to be considered as a class by themselves—so to speak, self-supporting—appears to sum up all that can be said on our present subject. Notwithstanding its importance, it is often overlooked by actuaries. Thus we sometimes hear the remark made, that "the claims of the year have fallen on large amounts." In this form of expression it is implied that a certain number of claims, large or small, was to be expected in the year; and that large claims have occurred where small ones were to have been expected; but there is clearly no reason for supposing that, because the claims among the small policies have been less numerous than might have been expected, therefore the claims among the large policies should be more numerous than the expectation. There is, in truth, no possible relation between the number of claims in the two classes of policies—large and small; and under the circumstances supposed, it would be more correct to say simply, that an unexpected number of deaths has occurred among the large policies.

So, again, I have heard it suggested that an Office might fix its maximum risk at such an amount, that if all the anticipated claims of the year were of the maximum amount, the total claims would just absorb a year's income. But this rule implies, that the more numerous the small policies are in an Office, the greater is the risk of each of the large policies becoming a claim, which is manifestly absurd. We conclude, on the contrary, that the amount of the maximum risk is to be fixed independently of the income or assets of the Office, by reference to the number of policies of maximum amount which it is probable will be effected—subject to the modification stated previously, when the income of the Office is *very* large.

I have spoken hitherto of the division of insurances simply into large and small; but, in practice, there will always be insurances of intermediate amount, more or less numerous; and in applying our rule, it will not be necessary to confine our attention to the insurances of the precise maximum amount, but we may fairly class with them such as do not much fall short of that amount. Thus, if the premiums are loaded 25 per cent., so that the Office premium for £4,000 is equal to the net premium for £5,000, we may fairly group together all the insurances for £4,000, £5,000, and intermediate amounts; and so with the insurances from £8,000 to £10,000; so that if the maximum of an Office is £5,000, the position of the Office will be considered satisfactory, if the number of insurances ranging from £4,000 to £5,000 is sufficient to give an average.

With regard to the particular limit to be fixed by an Office to its risks, it may be suggested that if the Office wishes to keep more than £5,000 it is scarcely worth while to stop short of £10,000, as the insurances of intermediate amounts are not very numerous. There is much weight in this remark, when it is the practice of the Office to retain the whole of the risk under all the policies it grants; but the practice of "reassurance," by which Offices interchange risks the one with the other, enables an Office to fix on any particular limit that may for the time being seem advisable. In the early years of an Office, it will seldom be advisable to fix a higher limit than £2,000 or £3,000; and as the number of large cases increases, the limit can be increased by steps of £1,000 to £5,000, £6,000 . . . £10,000, and even further, if justified by circumstances. The Office, while retaining only £2,000 at its own risk, may legitimately issue its policy for £5,000, and "reinsure" the balance, £3,000, with other Offices, which may be expected to make it a return. In this way, while the large risks of the Office are few, its position is greatly improved, since the probability of the occurrence of an inconvenient loss is greatly diminished.

As far as I can learn, no Insurance Company at the present time retains more than £15,000 at risk upon a single life; and I am only aware of one Office which retains so large an amount. But, if the arguments given above are valid, there would seem to be no objection to an Office keeping £20,000, £25,000, or £30,000 on one life, provided it can get a sufficient number of similar insurances, *i.e.*, not less than at least six or seven each year.

I now pass on to the consideration of various questions more or less connected with our general subject.

I have first to remark, that when a policy is effected for the whole term of life, the policy usually has a "value," which increases from year to year; and when we examine the effect this fact produces on the financial course of an Office, we see that in the first year of an insurance any claim arising must be paid out of the premiums received in the year; but in every subsequent year there is a reserve appertaining to each policy, which is available towards payment of the claim when it arises. We may say, then, that when a claim arises in a year subsequent to the first, a part of the sum assured—equal to the value of the policy—is paid out of the accumulated funds; and the balance only is paid out of the income—premiums and interest. The loss of the Office is confined to the latter amount; the payment as regards the value of the policy being more in the nature of the repayment of a loan. Thus, then, the risk of the Office decreases year by year, being only equal to the excess of the sum assured over the value of the policy. It follows that when a life has been for some time insured for the maximum amount in an Office, a further insurance may be accepted for a sum equal to the value of the existing policy, without increasing the risk of the Office beyond the usual maximum.

In illustration of this principle, let us suppose that an Office whose usual maximum is £5,000, has, through an oversight, been for some years exposed to a risk of double that amount on a particular life. Thus, a policy for £5,000 may have been granted on the life of James Duff, M.P., and another policy for £5,000 may subsequently be granted on the life of the Earl of Fife; and several years may elapse before it is observed that these are different names of the same person. When at last the discovery is made, what sum should be reinsured in order to reduce the risk of the Office to its usual maximum? If V_1 and V_2 are the values of the two policies, we see, by applying the principles laid down above, that the actual risk of the Office is £$10,000-V_1-V_2$; so that the sum to be reinsured will be £$5,000-V_1-V_2$.

Next, what is the risk of an Office under a *reversionary annuity*; as, for example, when an annuity of £1,000 is assured to be paid to x for life after the death of y? Here, if y die during the first year, the Office becomes liable for an annuity of value $1,000\times a_x$; and this therefore must be considered as the amount of the risk in the first year. The amount of the risk in the second year is similarly $1,000\times a_{x+1}$, which is less than the risk in the first; and the risk becomes less each year that the policy is in force. If such a policy is effected by means of a uniform annual premium, it will be found that the policy has no value, or a negative one, at the end of any number of years from the date of the assurance.

It may be suggested that in this case the usual maximum may be properly exceeded, because the Office has not, in the event of a claim, to pay a large sum down, but only to make the various payments of the annuity as they fall due. But this reasoning appears to me inadmissible; for we must consider what is the amount of the loss which would have to be provided for in a valuation; and the amount of the sum payable at the time of the claim is a matter of very secondary importance. I therefore conclude that any Office would act unwisely if it retained a larger risk under a reversionary annuity than it would do under an ordinary policy.

These examples may suffice to show that the risk of an Office, or the amount which it is liable to lose in the event of death, is often a very different thing from the payment it will be called upon to make. But this distinction is often lost sight of. Thus, it is the practice of some Offices to retain a larger sum at risk under a policy that is effected without profits, than is retained under a participating policy; on the ground that the bonuses may considerably increase the sum payable at the time when the claim arises. But if, as before, we consider the "risk" of the Office, which is only the excess of the sum assured and the bonus over the value of the same, we see that in practice it can seldom happen, however large the bonus may be, that the risk of the Office shall exceed the original sum assured; and we therefore conclude that there is no sufficient reason for retaining a less sum at risk on a bonus policy than on a non-participating policy.

From the same considerations it follows that when an insurance is effected on the last of two or three lives, the Office may properly retain at its own risk a larger amount than it would retain on a single life. At first sight we might suppose that a double sum assured could properly be retained on the last of two lives. "For," it may be said, "in the case of an insurance for double the usual maximum on the last of two lives, the risk is divided between the two, and the risk on either life is only equal to that usually undertaken by the Office. Thus, when one of the lives dies, although no cash payment has to be made by the Office, yet a loss has occurred, which has to be provided for in a valuation; and when the second life dies, although a double sum becomes payable, yet one half has been already provided for; and the loss, properly so called, consequent on the death, is no larger than in the case of an insurance of the maximum amount on a single life." But when we examine more closely, we shall find that this reasoning is only correct on the supposition that the premium for an insurance on the last of two lives is one half that for an insurance on one of the lives. For, in general, the risk is not *equally* divided between the two lives; but when one of the lives dies, the consequent loss is less than half the sum assured. For example, suppose that an insurance for £10,000 is granted on the last survivor of two lives, each 40 years of age, at a premium of £2 per £100—the premium for an insurance on a single life of 40 being £3 per £100; then, if one of the lives dies in the first year of the insurance, what is the consequent loss sustained by the Office? The annual premium, £200, would insure a sum of £6,666. 13*s.* 4*d.* on the remaining life; but the sum now payable on the dropping of that life is £10,000, so that we may say that £6,666. 13*s.* 4*d.* is provided for by the premium receivable, but the remaining £3,333. 6*s.* 8*d.* is unprovided for; and is, in fact, the loss which has occurred in consequence of the failure of the first life. In this case, then, the Office would lose £3,333. 6*s.* 8*d.* by the death of the first life, instead of £5,000 as assumed in the foregoing reasoning; and would then have £6,666. 13*s.* 4*d.* at risk on the second life, its usual maximum being £5,000. We shall determine the amount which may be properly insured on the last of the two lives as follows:—The annual premium on an insurance of £5,000 on one of the lives is £150, which will insure £7,500 on the last of the two lives; and that therefore is the sum which can be insured on the last of two lives of the age of 40, without exposing the Office to a larger risk than £5,000 on a single life.

Next, suppose that there are two lives of 25 and 50; that the premiums at those ages are £2 per £100, and £4 per £100; and that the premium for an insurance on the last of 25 and 50 is £1. 15*s.* per £100. As before, suppose that £10,000 is insured on the last of 25 and 50, at an annual premium of £175. This premium will insure £8,750 on a life of 25, and £4,375 on a life of 50. Hence we see that if 50 dies in the first year of the insurance, the risk left depending on 25 is £8,750, and the loss is only £1,250; while if 25 die, the sum depending on 50 is £4,375, and the loss is £5,625. To get the sum that may be insured properly on the last of 25 and 50, we take the premium for £5,000 on a life of 25 (*i.e.*, £100), and since this will insure £5,714 on the last survivor of 25 and 50, that is the largest sum that ought to be insured on the last of the two lives. If x and y are the two lives, of which x is the younger, π_x the annual premium for an insurance of £1 on x, $\pi_{\overline{x.y}}$ the premium for the same insurance on the last of x and y; then the amount which may be insured on the last of the two lives is $\frac{\pi_x}{\pi_{\overline{x.y}}} \times S$, where S is the maximum sum insured on a single life.

If, however, the lives are closely related, the risks cannot be considered as wholly independent; and it would scarcely be prudent to incur the risk of the larger sum on the last survivor of two closely related persons.

So again, we see that an Office might grant an "endowment" to be payable only in case a nominated person attain an assigned age, of a much larger amount than any insurance it would undertake; for the possible loss, in consequence of the life attaining the assigned age, will be only a fraction of the sum assured.

Another practical question arising in the course of life insurance business is this—Shall an Office take the same amount of risk in an old life as it takes in a young one? This question is to be answered in conformity with the principles we have laid down, by reference to the comparative number of policies on the lives of old and young persons. But it must be borne in mind, that we need not restrict the comparison to the insurances effected on young and old lives; for we may group together all the policies existing on old lives, whatever their duration. If, then, we should find that the insurances on the lives of old persons are not sufficiently numerous to form an average among themselves, it may, perhaps, be prudent to insure a smaller sum on the life of an old person than on a young one. It must be considered, on the other hand, that the test of medical examination is much more efficacious at the older ages than at the younger. It must also be remembered that since the premium on an old life is higher than on a young one, the number of old lives sufficient to give an average is less than the number of young lives required for the same purpose. In actual practice, we may probably say that there is no sufficient reason for insuring a less sum on the life of an old man than on the life of a young man.

This question has been considered by Mr. Scratchley, who has given a different answer to it. He says, in his *Treatise on Life Assurance Societies* (p. 36, App.)—"If £3,000 be the amount of policy granted on a life aged 30, not more than £1,677 should be granted on a life aged 60, otherwise the Society would experience *moral* disadvantage; for it is obvious that although from the premiums being larger on policies in advanced ages, the value of the *mathematical* expectation of the ultimate payments received by a Company from an assurance on an old life may be equal to that from a corresponding assurance on a younger and consequently longer enduring life, yet the *moral* expectations of the immediate risks for any current period can only be rendered equal to each other, or nearly so, by a large average of about the same age, or by some system of graduation of the amount assured, so as to tally with the increase of immediate risk attending the fact of the ages being older." Accordingly, he gives a table of the amounts to be insured at various ages. The above reasoning, however, appears to be erroneous, inasmuch as it is independent of the *relative* numbers insured at the various ages. If the number of persons insuring at the age of 60 were equal to those insuring at 30, I have no hesitation in saying, that the Office might safely retain a *larger* sum at risk on the old lives than on the young.

With regard to the so-called "moral" expectation, mentioned by Mr. Scratchley above, I have to confess that I am unable to see that the theories on this subject are of the least practical utility. I have preferred, throughout this paper, to treat the various questions considered independently of those theories; and I trust that my conclusions will be found none the less satisfactory on this account.

I have now only to remark, in conclusion, that I am well aware that my subject is far from exhausted by the preceding remarks, and that much remains to be done to complete the mathematical part of the inquiry. I trust, however, that this attempt to elucidate some questions hitherto very little discussed, will be found of some practical utility to those engaged in the management of Life Insurance Societies.

General Average. *By* RICHARD MORRISON.

(Continued from page 361, vol. xii.)

IN considering the various expenses, or, as some writers term them, expenditures, that are the results of a general average act, the same test must be applied as in the case of sacrifices. Were they had recourse to as being, in the exercise of sound judgment and prudence, indispensably necessary for procuring the safety of all concerned in the adventure? Might the act from which they proceeded be the means of arresting any course of events which, if allowed to take their full scope, would terminate in the destruction of the vessel and the consequent loss of the freight and cargo? It is true human judgment is likely to err under these, as in other circumstances of like nature, and expenses may be incurred needlessly, for the ultimate consequences of an act cannot be always perceived; but if the end proposed will admit of no other course by which the vessel and the property in her may be preserved at the time, and if the measures resorted to are such as would not be employed in the ordinary prosecution of the voyage, there is reason for admitting such as a lawful subject of contribution. When common sense is exercised, and the circumstances are regarded comprehensively and clearly, the results are generally found to justify the means adopted in cases of danger needing prompt action.

Although in most works on average much attention has been devoted to the subject of jettison, and sacrifices generally, as being the most ancient sources of contribution, undoubtedly the more important act is comprehended in the term *expenses*. It is by far the most common, considering the many general average acts which take place. Many distinctions might be drawn between the two. Generally the jettison of a comparatively small portion of the cargo suffices to relieve the vessel; or a sacrifice of part of the ship's furniture can be sometimes replaced at once, as vessels are usually supplied with two or three sets of sails and extra spars, to be substituted for those which may be lost or cut away. In such a case the act is attended by the actual loss of the property sacrificed; it could be replaced at the market value when the vessel arrives at her destination, so that under ordinary circumstances no great pecuniary loss interferes immediately with the prosecution of the voyage. But with expenses the case may be very different; for if a ship, by being disabled by a tempest, or through springing a leak, is compelled to put into an intermediate port for repairs, much money may be expended, and great difficulty experienced in raising funds to defray the charges incurred. Under such circumstances the actual pecuniary loss may be considerably augmented by the high rate of interest for the advance of money, by commissions, or by the loss by sale of part of the cargo to pay the expenses; all of which are chargeable *pro ratâ* to the interests benefited thereby. And when high percentages are thus charged, or the cargo sold does not realize even its prime cost, owing to the intermediate port not offering the advantage of a market for the merchandize disposed of, it will be evident that although both sacrifices and expenditures for the common benefit are rightly denominated subjects of general average, the positive loss in many instances admits of some distinction. With loss by sacrifice the act and its consequences can be estimated with tolerable accuracy when the interests become dissociated; whilst with expenditures, if we regard contingencies, it cannot be foreseen, when a ship enters a port, what charges may have to be incurred before she again proceeds on her voyage, nor what difficulty may be experienced in raising money to meet the disbursements, by which additional loss may be incurred by all parties concerned. It is quite true the act of jettison or sacrifice might happen before the ship is compelled to bear up to repair damage—the one might necessitate the other—in which case equal loss would be entailed upon all parties; but the two not being necessarily associated, and being most frequently distinct from each other, these remarks apply to them as specified. It is true also that the act of jettison is sometimes attended by another loss, and that the sacrifice of spars, &c., may occasion other damage; but as a difference of opinion exists as to the propriety of allowing these as general average, we have here considered the act according to the present "practice of Lloyd's."*

There is likewise a most important difference between the two great features of general average. When an *expense* is not allowed as a subject for contribution, it can be recovered from either the owner of the cargo or the freight, or it falls upon the shipowner as such; but a *loss* or *sacrifice*, whether pertaining to ship, freight, or cargo, if it be not recoverable as general average, falls on the interest to which it applies, or its representatives as insurers, exclusively. This gives rise to many disputes and complications, for opinions frequently differ, as to the propriety of charging the sums which are disbursed, professedly for the general benefit, but which, upon close examination, are found to apply specifically to one or the other of the interests conjoined.

At the beginning of the present century it was held by the common law, as far as such cases came under its consideration, that *all* the charges incurred by a vessel which put into a port in distress, should be made good by a general contribution; and the foreign laws were to the same effect. Gradually, however, the questions which arose concerning these expenses were more fully considered, and opinions were expressed that some of them were not matters of general average, as they were not incurred expressly for the common benefit. It was said that if the cargo was specially benefited or preserved by them, such expenses should fall upon the owners of it; and those disbursements which were made in order that the vessel might continue her voyage (such as outward charges from a port of refuge), should be chargeable to the owners of the freight, as they were held to be for its special advantage. It has been likewise held, that when a vessel puts into a port merely in consequence of contrary winds, the expenses incurred at that time are to fall exclusively upon the shipowner.

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* In support of the above a case is subjoined as an illustration:—A Spanish ship from Manilla to Liverpool put into St. Helena, being in a leaky condition, and for repairs generally. No previous general average act had taken place. The cargo was, by order of surveyors, gradually discharged; and when they had ascertained the full extent of the damage, she was condemned as unseaworthy and sold. The agents required large sums of money from time to time, to meet the expenses of ship and cargo. Their claim for advances exceeded £7,800; and the master, after vainly endeavouring to raise money by hypothecation of the cargo to be shipped to its destination in another vessel, sold part of it at auction for £6,500. The enormous sum advanced was partly made up of £1,800 for warehouse rent of the cargo; £450 was charged as commission for trouble and taking care of the cargo, £1,000 for landing, add £1,000 for reshipping it. The total amount of the agents' charge at St. Helena (including the above), apart from the other expenses for which they advanced money, was £4,800! When the statement was adjusted in England, about £2,200 was apportioned as general average, so that a large proportion of the charges fell on the cargo and freight. And as the contributary values were as follows—ship £895 (what she sold for), freight £2,473, and cargo £21,027—it will be seen how little the ship (the condition of which caused the act entailing the expenses) had to pay, in comparison with the other interests. This is an extreme case, but it illustrates the principle stated, and is but one out of a great number that occur in the course of the year.

The chief expense of a general average act necessitating recompense, and to which, it has been said, all such acts are in principle analogous, is salvage. It must be remembered that when this term is used with reference to nautical affairs, it signifies the act of saving rather than the property saved: an allowance or compensation for exertions made to save the ship from danger, arising from the sea, fire, or pirates. It is seldom that it is employed in the other sense. Where the word "services" is affixed, the distinction is at once indicated.

The most frequent case of salvage arises from a vessel becoming disabled through the violence of the winds and sea, and requiring to put into a port of refuge to repair or refit. Where it is practicable, the aid of a steamer is obtained to tow her; and where that is not at hand, fishing boats are employed if met with. Sometimes men from the shore are taken on board to pump if the vessel be leaky, or to assist in working the ship if the crew have become exhausted. As a necessary consequence of bearing up for a port, pilotage in, and port and harbour dues, have to be paid. They are charged as general average.

It sometimes occurs that the master is disinclined to adopt measures so necessary for the preservation of the interests endangered, as the salvors are apt to magnify the danger of the coast, that they may demand of him large sums for their services; requiring him, perhaps, to sign an agreement before they board the vessel. A prudent man in such a case stipulates that the claim shall be referred to proper adjudicators, and both parties bind themselves to adhere to their arbitration. Disputes frequently arise which arbitrators cannot settle, but which are ultimately decided by the courts of law appointed for such purposes; whilst the number of the cases under consideration afford ample proof of salvage services being fruitful causes of general average contribution. Assistance is generally needed where vessels accidentally get aground; which may arise either from ignorance of the coast on the part of the master, or from the prevalence of thick foggy weather, such as is so often experienced in the Baltic trade, even during the summer months.

It has been previously stated that under certain circumstances vessels may be compelled to slip from their anchors, which act entails a loss to be made good in general average. It frequently gives rise to another proceeding—one which is a common source of claim for salvage services. If the anchor and cable be sacrificed to avoid being driven on shore or to avoid a collision, and the vessel should run for the nearest port, the services of fishing boats or other craft are sometimes accepted in order to recover them. Although such efforts may not be attended with success, the services so rendered are allowed; as are the expenses of carrying off anchors and cables to supply the place of those sacrificed under similar circumstances, where the vessel cannot enter the harbour, from being either too deeply laden or from the violence of the wind and sea. If they should be picked up after others have been purchased and carried on board, they are usually sold by the agent of the vessel at that port, for the benefit of those concerned; and when the salvage expenses are paid, the net proceeds are deducted from the cost of the new ones.*

There is one expense for salvage services which possessed much interest to the last generation, but which has seldom been made of late years in connection with British shipping—viz., money paid to men-of-war or privateers for recapture from an enemy. In the time of George III. the salvage fixed by law was one-eighth of the value of the property to the king's ships and one-sixth to privateers. The rule in France at that time was to give a third of the value of the property saved to the recaptors. As in the event of war such services might again form subjects of contribution, the facts are mentioned incidentally.

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* During the winter months the fishermen of the eastern and southern coasts of England obtain their chief maintenance from these services. It is of the highest importance that a ship should never be without a good anchor, often almost literally "the sailor's last hope"; and these men knowing its value, sometimes demand large sums (especially if the danger of boarding the ship be great), which in the end are settled only by arbitration or legal decision. From a report of the Lloyd's special agent at Deal, it appears that nine cases were referred to him after the gales of the 7th to 12th January of the present year. One was a claim for £250 for taking off an anchor and chain at midnight. It was subsequently reduced to £150.

The expenses which fall specifically on the cargo are warehouse rent, and all charges incurred either for its preservation or restoration. As the general average act does not cease till the cargo is in a place of safety, all the expenses of landing it, and carriage to that place, are general average.

To the freight are charged the expenses connected with the reshipment, such as labour, reloading and restowing the cargo, and all outward dues on the ship; also steamer where engaged to tow the vessel out to sea.

Notarial expenses are general, as are commissions for trouble and attendance while the ship is under average. All others are apportioned *pro ratâ*. Surveys are charged either to the ship or cargo, according to that on which it was held. The *first* survey, if it be for the purpose of ordering the discharge of the ship, to ascertain the extent of the damage or the necessity of repairs, is placed to the general column, as it pertains to the collective interests.

Mention was made at the commencement of this article, of the proceedings of the International General Average Congress, and of the rules framed, which were embodied in the "Draft of a Bill" as a basis for legislation. Although legislators and mercantile men generally will fully endorse the opinion expressed by one of the Presidents, "that it is most desirable, if practicable, that one uniform rule should prevail among maritime communities in a matter in which seafaring men are continually brought into collision with foreign laws," it is scarcely to be expected that the matter will be actively promoted. Those who are most interested in it, and who therefore ought to be looked to in the first instance in obtaining the desired end, are for the most part indifferent. Many of them protect themselves against the consequences of foreign usages which admit many charges as general average which we discard, by the insertion of clauses in the policies they take out, by which they obtain an indemnity for any charges for which they become liable by those usages. Most policies contain the following or similar written clauses:—"General average recoverable according to foreign statement if so paid by the assured." But it would be most desirable and equitable that insurers should not be the sufferers from anomalous practices, the burden of which, if felt solely by assured, might induce the latter to use every exertion to obtain an adjustment whereby many existing inconveniences might be obviated. The House of Commons generally includes a number of intelligent and active members of the wealthy class of merchants and shipowners, who, doubtless, by their public and private influence, might obtain the support of those in office; for the only successful plan would be to make it a Government measure, according to the suggestion offered at the Congress in 1864.

In the year 1747 an attempt was made to procure an Act of Parliament for the benefit of the shipping interests. Leave was given to bring in the Bill, and a committee was appointed to prepare it; but the matter ended with that proceeding, for it never became law. The Act proposed differs materially from that of 1747. It refers entirely to the subject of general average; the other related almost exclusively to "the better regulating of assurances on ships, and on goods laden thereon;" and resembled, in some respects, the codes established in foreign countries.

Should the attempt be again made to legislate upon a question, which has, nevertheless, some affinity to it—both having the interests of the maritime communities as the object in view—it would be a matter of regret if, after the lapse of nearly 120 years, no better results were attained; particularly as the prejudice which existed in the one case does not apply to the other. When we consider how little was known respecting insurance and average at that time, that there were but two or three small works on marine law in use in England, and that since that period there have been most able digests and treatises written in comparatively large numbers; and when we consider the immense growth of commerce (especially of Great Britain and America), which has necessarily given rise to numerous cases which have been decided by *usage* or *law*—decisions which have gradually formed principles, not perhaps applicable to every case that may arise, but which suffice to influence most essentially the conduct of business; when we consider also that legislation, in the main, is the outgrowth and evidence of the enlightenment and prosperity of a nation; and further, that many nations have united to obtain a successful result; it may be thought that the time has come when endeavours should be made to procure uniformity of practice with regard to general average acts, and that the labours of those who have laid the foundation of the work should not become absolutely nugatory.

One of the disputed questions to which attention was directed, respects the damage done to the cargo by opening the ship's hatches to effect a jettison. A similar occurrence is the placing of part of the cargo on deck, which may be washed overboard or damaged, during the jettison of merchandize which it might be desirable to cast overboard in preference to that so placed, on account of its weight, or from some other cause rendering its removal necessary. Now, it would seem that it could scarcely be questioned whether it ought to be allowed, or not. It would appear to be a part of the act of jettison in both instances, "as the accessory follows its principal" (Arnould). When a heavy sea is washing over the decks, it stands to reason that the hatches could not be opened to perform an act to benefit all concerned, without the water getting into the hold, and consequent damage occurring. It is necessary in the finest weather to have them caulked, and made water-tight by tarpaulins. The same reasoning applies to the case of goods placed on deck to get at others below: when the ship is rolling and lurching about, under some circumstances it would be almost impossible to prevent them from falling overboard, much less to keep them from getting damaged. All writers agree that such damage ought to be allowed as a claim for contribution; yet, in practice, it is not admitted. The reason of this will be more concisely and clearly given in the words of the Report to the Liverpool Chamber of Commerce, October, 1864:—"It is not held to be inadmissible in principle . . . . but a nearer approximation to *practical* justice is attained by excluding it than by admitting it, seeing the temptations which it offers in many cases to fraud, and the ground that it furnishes in all to disputes." It then states that the exclusion "does not often run counter to actual equity"; for, generally, but little damage is sustained in that way; that if much water were likely to get down the hold, it would neutralize the object proposed by the jettison, and the hatches would not then be opened, but goods taken through the cabins and bulkheads; and that where the ship has been labouring much, it is hardly possible to distinguish between the damage caused by the water in the vessel and that which would wash into it. This is the substance of the argument, including one other reason, perhaps the most cogent of all; the importance of which is still more enhanced when we consider the knowledge of such facts, possessed by the gentleman who makes it.* "In practice, claims of this nature are made generally when goods are insured 'free of particular average,' and not when they are insured against all risks—a fact which by itself raises a presumption against the correctness or expediency of admitting such damage."

These observations do not quite agree with the preceding remarks; but the writer conceives that the two last-mentioned considerations explain the discrepancy, namely — the impossibility, in many instances, of distinguishing between the two causes of damage; and the fact that claims *are sometimes* made for damage under the circumstances in question, which would not be the case without some evidence being submitted in proof, and that not merely inferential or supposititious. The reason suggested, that not much damage results, may arise from the frequency of the claim being made as *particular average*—which requires little other evidence than that it exists, as the insurer pays for the sea damage; and the argument that goods might be taken from the cabins would scarcely be admitted by nautical men, for the merchandize placed in that part of the ship (to get at which would be extremely difficult in almost every instance) would not be, in most cases, such as would be likely to relieve her, the dead weight being placed as much as possible below the water-line and in the centre of the vessel. Again, though it might not be possible during a jettison to prevent the water rushing into the hold, it would hardly rush down in such volumes as to neutralize the act; and even supposing it did so, it would have only a temporary effect; for the ship would be *lightened* as the water could be pumped out, and the desired object attained, which object was the speedy relief of the vessel.

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* Mr. Rd. Lowndes, of Liverpool.

Although it was finally decided to allow it as general average when the jettison was so allowed, we know of cases where adjustments so made were restated.

This case has been cited, from many which might be adduced, chiefly to show the discrepancy which exists—and which it is so difficult to remove—between *principle* and *practice*, as pertaining to general average losses.

The hatches may be opened for another purpose, namely, to extinguish a fire that has broken out amongst the cargo; which act might result in damage to other portions of the cargo not endangering the safety of the vessel. Or, damage may be done to both ship and cargo, when under such circumstances the decks are "scuttled"—*i.e.*, when holes are cut in them to let down water into the hold with the same object. Abroad this is allowed as a subject of general contribution, but in England it has been hitherto excluded, although the propriety of the rule has been frequently questioned. The same objections to its exclusion are thought not to exist, as in the previous case; and in effect, should it be allowed as general average in accordance with the rules adopted, the damage resulting from such an act would be more equitably adjusted than according to the existing practice.

But the subject on which the greatest difference of opinion has ever existed, has reference to the allowance of the wages and provisions of the crew of a vessel which has been compelled to put into an intermediate port for repairs, or from any other cause necessitating the entering a port of distress.*

The expense of repairing the vessel under such circumstances is by many countries allowed as general average. Without dwelling on the point, it may be stated that in England such repairs are always disallowed.

The wages and provisions are not treated as general average in this country. The reason assigned for their exclusion is, that the shipowner is bound by the terms of his contract with the proprietors of the cargo to keep his vessel well manned and provisioned till the voyage is completed and the contract at an end. From the very nature of the relation in which he stands to those parties, he must expect that such contingencies may arise, and that the crew will have to be maintained through all the circumstances of the voyage, of which the repairs at a port of refuge form a part. It is by no means such an uncommon event that a shipowner could expect to be entirely exempt from it; and the possibility of such delays protracting the voyage ought to be considered when he fixed the rate of freight to be charged.

In the United States the wages and provisions are invariably charged as general average, and on the continent they are usually admitted as such. But it is not merely whilst the vessel is in the port of refuge that the advocates for this rule seek its universal admission. They urge that it should obtain from the time when the ship, being in distress, alters her course to seek a port of safety, and until she has refitted and continues her voyage. This is the general practice adopted abroad. Formerly they attempted to include also unjust detention and embargo under the same rule, but these have never been allowed either in England or America.

The International Congress resolved to conform to foreign usage, and to allow these expenses as general average. It was represented, that "when a general average arises the shipowner is the only person who incurs pecuniary expenses and does not recover them; that this is a great grievance and a strong temptation to evade the practice of this country, which rejects those expenses." This, we believe, is accomplished, either by having the statement made in some American port, should the ship be bound to one, or by insisting on the insertion of the "foreign clause." It has been said that the outward trade of this country is not less than half of the whole trade; consequently, that it is not unreasonable to hold that half the trade is, as regards general average, governed by foreign statements; for most merchants, as has been previously mentioned, insist upon their goods being insured with that clause, so that they may recover from their underwriters the full amount for which they become liable by the statement being adjusted at a foreign port. In consequence of this, insurers, by taking the risk, admit the crew's wages and provisions, not merely for the time during which the vessel is in the port of refuge (which is all they propose to allow), but from the time the ship bears up for that port, which in many cases is a material point. The words of the section are— . . . "from the time of entering such port until the ship shall have been made ready to proceed upon her voyage."

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* Concerning these, Sir Joseph Arnould observes—"There is hardly any point, even in the perplexed doctrine of general average, in which there is such a great diversity in the positive laws of mercantile states" (vol ii., p. 911).

Although no doubt this is the correct view to take of the matter, if we regard the anomalies connected with it; it is, nevertheless, one which, according to some, admits of some show of reason on the other side. It is said by those who object to the allowance of these expenses, that the proprietors of perishable cargoes suffer by the delay, and that if the shipowner receives compensation for the crew's detention, they ought to claim for the deterioration. But then, if the ship were not repaired, on this reasoning the goods would, if perishable, soon become absolutely valueless; so that by the proceeding which enables her to continue the voyage they are directly benefited. This will appear evident in the case of a cargo of fruit, or grain which had become heated. And even in these extreme cases, if the condition of the cargo should be such, that any delay in the voyage would tend to its deterioration to any great extent, it would be sold at the port where the repairs were effected; and although a loss might, and most probably would, be the result, yet the act of reparation would be as much for the benefit of the cargo as for the ship; perhaps more so, for if she kept the sea and strove to reach her destination in her disabled condition, when she arrived there the cargo might no longer exist in specie. Besides, as the shipowner would not get the repairs allowed as general average, and as workmen's labour is charged at so much per day in many cases, the master would have every motive for using all reasonable dispatch in getting his vessel ready for sea. As also the owners would have to pay their proportion of the wages and maintenance of the crew, by which the profits of a partially unsuccessful voyage would be still further lessened, it seems only just that the burden should be more equitably distributed than according to the practice hitherto adopted. This view of the matter will apply likewise to another objection—viz., that the allowance of these expenses would encourage long detention at ports of refuge.

A case of much importance is that of voluntary stranding, or the running a vessel on shore to escape a greater peril. Ought the repair of the damage to the ship that may be the result of such an act, or the damage done to the cargo, to be allowed as general average? Several countries of Europe and the United States admit it. The rule in England is not to allow it. Mr. Baily considers that the rule is good in the generality of cases, because the extent of the damage done by that act cannot be ascertained, and other damage must have existed previous to it, or it would not take place. But "where the facts are not disputed, it is inequitable, and might be overthrown by a legal decision." He quotes a remark of Chief Baron Pollock, "Do not exclude the truth where accessible, because you cannot always obtain it." Mr. Baily, whose work carries weight, both on account of his mode of treating the subject and the practical knowledge he possesses, is clearly against any alteration on the point; although he asserts that the defence of the rule "rests more on policy and expediency than on equity." He admits that cases might arise where the voluntary stranding might not be attended by loss or damage to the cargo, but necessarily to the ship; and supports this view by the case of a vessel run ashore to avoid capture, when the rule would be manifestly unjust; but considering the rareness of such occurrences where only the cargo is uninjured, it is expedient to adhere to the present practice.

Where the motive for running a ship on shore is to prevent her from foundering, it is not likely that the cargo will have escaped contact with water; for the presence of water in the hold is the cause of the act taking place. In such a case it would be practically impossible to distinguish between the two causes of damage.

Such is the gist of his reasoning; from which it appears that in principle it ought to be admitted, but from the complicated circumstances connected with the act, it ought to be excluded; and on account of the "endless disputes and practical injustice" which would result from any alteration.*

It appears to have been the opinion of the most eminent practitioners of the bar, about the beginning of the present century, that cases of voluntary stranding should be considered as subjects of general average contribution, where the ship was got off in a condition to prosecute her voyage; under which circumstances, "the *damage sustained,* and the expenses incurred, would be a partial loss of the nature of a general average." The question has, however, never been brought into our courts of law; and, indeed, it is seldom that such an occurrence takes place; for, as a high authority has said, unless there is some hope of the ship being saved, it is merely *sauve qui peut.*

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* See Baily, pp. 42, 75, 82.

(*To be continued.*)

A Budget of Paradoxes. *By* PROFESSOR DE MORGAN.

(Continued from p. 301, vol. xii.)

No. XV. 1836—1839.

REFERRING to the hoax about Mr. Goulburn in No. XIV., Messrs. C. H. and Thompson Cooper have corrected an error, by stating that the election which gave rise to the hoax was that in which Messrs. Goulburn and Yates Peel defeated Lord Palmerston and Mr. Cavendish. They add that Mr. Gunning, the well-known Esquire Bedell of the University, attributed the hoax to the late Rev. R. Sheepshanks, to whom, they state, are also attributed certain clever fictitious biographies—of public men, as I understand it—which were palmed upon the editor of the *Cambridge Chronicle*, who never suspected their genuineness to the day of his death. Being in most confidential intercourse with Mr. Sheepshanks, both at the time and all the rest of his life (twenty-five years), and never having heard him allude to any such things—which were not in his line, though he had satirical power of quite another kind—I feel satisfied he had nothing to do with them. I may add that others, his nearest friends, and also members of his family, never heard him allude to these hoaxes as their author, and disbelieve his authorship as much as I do myself. I say this not as imputing any blame to the true author, such hoaxes being fair election jokes in all time, but merely to put the saddle off the wrong horse, and to give one more instance of the insecurity of imputed authorship. Had Mr. Sheepshanks ever told me that he had perpetrated the hoax, I should have had no hesitation in giving it to him. I consider all clever election squibs, free from bitterness and personal imputation, as giving the multitude good channels for the vent of feelings which but for them would certainly find bad ones.

Recueil de ma vie, mes ouvrages et mes pensées. Par Thomas Ignace Marie Forster. Brussels, 1836, 12mo.

Mr. Forster, an Englishman settled at Bruges, was an observer in many subjects, but especially in meteorology. He communicated to the Astronomical Society, in 1848, the information that, in the registers kept by his grandfather, his father, and himself, beginning in 1767, new moon on Saturday was followed, nineteen times out of twenty, by twenty days of rain and wind. This statement being published in the *Athenæum*, a cluster of correspondents averred that the belief is common among seamen, in all parts of the world, and among landsmen too. Some one quoted a distich—

"Saturday's moon and Sunday's full
Never were fine and never *wull*."

Mr. Forster did not say he was aware of the proverbial character of the phenomenon. He was a very eccentric man. He treated his dogs as friends, and buried them with ceremony. He quarrelled with the *curé* of his parish, who remarked that he could not take his dogs to heaven with him. I will go nowhere, said he, where I cannot take my dog. He was a sincere Catholic: but there is a point beyond which even churches have no influence.

Some account of the great astronomical discoveries lately made by Sir John Herschel at the Cape of Good Hope. Second edition. London, 12mo., 1836.

This is a curious hoax, evidently written by a person versed in astronomy and clever at introducing probable circumstances and undesigned coincidences. It first appeared in a newspaper. It makes Sir J. Herschel discover men, animals, &c., in the moon, of which much detail is given. There seems to have been a French edition, the original, and English editions in America, whence the work came into Britain: but whether the French was published in America or at Paris I do not know. There is no doubt that it was produced in the United States, by M. Nicollet, an astronomer, once of Paris, and a fugitive of some kind. About him I have heard two stories. First, that he fled to America with funds not his own, and that this book was a mere device to raise the wind. Secondly, that he was a *protégé* of Laplace, and of the Polignac party, and also an outspoken man. That after the revolution he was so obnoxious to the republican party that he judged it prudent to quit France; which he did in debt, leaving money for his creditors, but not enough, with M. Bouvard. In America he connected himself with an Assurance Office. The moon-story was written, and sent to France, chiefly with the intention of entrapping M. Arago, Nicollet's especial foe, into the belief of it. And those who narrate this version of the story wind up by saying that M. Arago *was* entrapped, and circulated the wonders through Paris, until a letter from Nicollet to M. Bouvard explained the hoax. I have no personal knowledge of either story: but as the poor man had to endure the first, it is but right that the second should be told with it.

The Weather Almanac for the Year 1838. By P. Murphy, Esq., M.N.S.

By M.N.S. is meant *Member of no Society.* This almanac bears on the title-page two recommendations. The *Morning Post* calls it one of the most important-if-true publications of our generation. The *Times* says—"If the basis of his theory prove sound, and its principles be sanctioned by a more extended experience, it is not too much to say that the importance of the discovery is equal to that of the longitude." Cautious journalist! Three times that of the longitude would have been too little to say. That the landsman might predict the weather of all the year, at its beginning, Jack would cheerfully give up astronomical longitude—*the* problem—altogether, and fall back on chronometers with the older Ls, lead, latitude, and look-out, applied to dead-reckoning. Mr. Murphy attempted to give the weather day by day: thus the first seven days of March bore Changeable; Rain; Rain; Rain-*wind*; Changeable; Fair; Changeable. To aim at such precision as to put a fair day between two changeable ones by weather theory was going very near the wind and weather too. Murphy opened the year with cold and frost; and the weather did the same. But Murphy, opposite to Saturday, January 20, put down "Fair, Probable lowest degree of winter temperature." When this Saturday came, it was not merely the probably coldest of 1838, but certainly the coldest of many consecutive years. Without knowing anything of Murphy, I felt it prudent to cover my nose with my glove as I walked the street at eight in the morning. The fortune of the Almanac was made. Nobody waited to see whether the future would dement the prophecy: the shop was beset in a manner which brought the police to keep order; and it was said that the Almanac for 1838 was a gain of £5,000 to the owners. It very soon appeared that this was only a lucky hit: the weather-prophet had a modified reputation for a few years; and is now no more heard of. A work of his will presently appear in the list.

Letter from Alexandria on the evidence of the practical application of the quadrature of the circle in the great pyramids of Gizeh. By H. C. Agnew, Esq. London, 1838, 4to.

Mr. Agnew detects proportions which he thinks were suggested by those of the circumference and diameter of a circle.

The creed of St. Athanasius proved by a mathematical parallel. Before you censure, condemn, or approve; read, examine, and understand. E. B. REVILO. London, 1839, 8vo.

This author really believed himself, and was in earnest. He is not the only person who has written nonsense by confounding the mathematical infinite (of quantity) with what speculators now more correctly express by the unlimited, the unconditioned, or the absolute. This tract is worth preserving, as the extreme case of a particular kind. The following is a specimen. Infinity being represented by ∞, as usual, and f, s, g being finite integers, the three Persons are denoted by ∞^f, $(m\infty)^s$, ∞^g, the finite fraction m representing human nature, as opposed to ∞. The clauses of the Creed are then given with their mathematical parallels. I extract a couple:—

But the Godhead of the Father, of the Son, and of the Holy Ghost, is all one: the glory equal, the Majesty co-eternal.	It has been shown that ∞^f, ∞^g, and $(m\infty)^s$, together, are but ∞, and that each is ∞, and any magnitude in existence represented by ∞ always was and always will be: for it cannot be made, or destroyed, and yet exists.
Equal to the Father, as touching his Godhead: and inferior to the Father, as touching his Manhood.	$(m\infty)^s$ is equal to ∞^f as touching ∞, but inferior to ∞^f as touching m: because m is not infinite.

I might have passed this over, as beneath even my present subject, but for the way in which I became acquainted with it. A bookseller, *not the publisher*, handed it to me over his counter: one who had published mathematical works. He said, with an air of important communication, "Have you seen *this*, Sir!" In reply, I recommended him to show it to my friend Mr. ——, for whom he had published mathematics. Educated men, used to books, and to the converse of learned men, look with mysterious wonder on such productions as this: for which reason I have made a quotation which many will judge had better have been omitted. But it would have been an imposition on the public if I were, omitting this and some other uses of the Bible and Common Prayer, to pretend that I had given a true picture of my school.

(*To be continued.*)

CORRESPONDENCE.

ON THE VALUE OF OPTIONS IN CERTAIN CONTRACTS.

To the Editor.

SIR,—I propose, in the present letter, to give the mathematical solution of the question which Mr. Stephenson enunciated in the April Number of the *Magazine* (p. 178), namely, to find the single premium P_x which a person aged x should pay to secure a deferred annuity of £1 to be entered upon at the age $x+n$ and to be continued for the remainder of life, on the understanding that P_x is to be returned at death, if that should happen during the n years, and that the individual is to have the option during the same period of withdrawing the sum P_x at any moment, and so putting an end to the contract.

Before giving the method of determining P_x, it may be as well to point out the reason why the formula which Mr. Stephenson has obtained is not the answer to the question proposed. I found it difficult, at first, to understand how that gentleman could have been led to believe that he *had* solved the question, seeing that throughout his solution the contingency of withdrawal is nowhere taken into consideration—but his mistake clearly lies in his imagining that because the *interest only* of P_x is used during the n years, therefore P_x is not *sunk* during that period, and consequently that it must be free to be withdrawn at any moment during the n years, or to be returned at death if that should happen in the same interval. An examination of his solution, however, will easily show that such an idea, as far as withdrawal is concerned, is quite erroneous. He says, if $p_{x\overline{n}|}$ be the annual premium payable at the end of the year for assuring to x a deferred annuity of £1 after n years, then $P_x \cdot \frac{i}{p_{x\overline{n}|}}$ will be the amount of the deferred annuity which can be assured by the conversion of the annual interest into an annual premium; but when he afterwards writes $\frac{a_{x\rceil n}}{a_x - a_{x\rceil n}}$ for $p_{x\overline{n}|}$, the assumption is involved that nothing but *death* will cause the payment of $p_{x\overline{n}|}$ to cease, and consequently $P_x \cdot \frac{i}{p_{x\overline{n}|}}$ can only represent the amount of the deferred annuity upon *that supposition*. This proves most conclusively that the contingency of withdrawal is not included in his solution. The reason why his formula gives a somewhat larger value for P_x than the ordinary expression, which provides merely for a return of the premium at the end of the year of death, is quite obvious, for his method of solution supposes the interest of P_x to be invested only at the *end* of each year, and therefore it leaves unappropriated the fractional part of the year's interest to the day of death, if death should happen at any time between the beginning and the end of one of the n years. His formula, therefore, is sufficient to provide for a return of P_x at death + the fractional part of the current year's interest to the day of death, and therefore if these sums were left with the Office till the end of the year of death the amount returnable would be $P_x(1+i)$, as Mr. Makeham has shown by a different mode of reasoning.

The mathematical solution of the question, which I will now give, is not without interest, and I will take the problem precisely as it is proposed, the premium being returnable at the moment of death, or withdrawable at any time in the n years, on demand.

Suppose it has been found from actual experience that in a large number, β', of such policies, the withdrawals in the first, second, third, &c., years have been w', w'', w''', &c., respectively, then, for any other number, l_x, of cases, the withdrawals would be $\frac{l_x}{\beta'}w'$, $\frac{l_x}{\beta'}w''$, &c., which we will denote by w_1, w_2, &c., and for the moment suppose £1 to be the single premium. Imagine l_x policies to be effected on the same day by persons aged x. Let δ_1, δ_2, δ_3, &c., be the tabular number of deaths in the first, second, &c., years of the existence of the policies, or the number of deaths that would happen if there were no withdrawals, and let λ_{n-1} denote the number of policies in existence at the end of $n-1$ years, that is, after deaths *and* withdrawals have taken place. Suppose w_n, the withdrawals in the nth year, to be distributed equally over the year, and the deaths δ_n to be so likewise; then, if t be any portion of the nth year, reckoned from its commencement, the deaths at the instant between t and $t+dt$ would, if there were no withdrawals, be $\frac{\delta_n}{l_{x+n-1}}\lambda_{n-1}.dt$, and the deaths at the same moment among the w_nt persons who have withdrawn since the beginning of the year would be $\frac{\delta_n}{l_{x+n-1}}w_nt.dt$; hence, subtracting the latter from the former, and integrating, we get

$$\frac{\delta_n}{l_{x+n-1}}\left(\lambda_{n-1}\int_0^1 dt-w_n\int_0^1 tdt\right)=\frac{\delta_n}{l_{x+n-1}}(\lambda_{n-1}-\tfrac{1}{2}w_n)$$

for the number of deaths during the nth year, in respect of each of which the Office would have to return the premium.

We thus get

$$\lambda_n=\lambda_{n-1}-\frac{\delta_n}{l_{x+n-1}}(\lambda_{n-1}-\tfrac{1}{2}w_n)-w_n \quad . \quad . \quad . \quad . \quad (1)$$

and by making n successively equal to 1, 2, 3, &c., the values of λ_1, λ_2, λ_3, &c., may be easily calculated.

Let t, as before, denote any portion of the nth year, and v^t the present value of £1, due at the end of the time t; then, at the commencement of the nth year the value of the risk of death or withdrawal taking place at the instant between t and $t+dt$ will be

$$\frac{\delta_n}{\lambda_{n-1}.l_{x+n-1}}(\lambda_{n-1}-w_nt)v^t.dt+\frac{w_n}{\lambda_{n-1}}v^t.dt.$$

If we integrate for the whole year, and multiply the result by $\frac{\lambda_{n-1}}{l_x}v^{n-1}$, we shall have for the value, at the time of the issue of the policy, of the risk of an individual death or withdrawal during the nth year, the expression

$$\frac{\delta_n v^{n-1}}{l_x.l_{x+n-1}}(\lambda_{n-1}\int_0^1 v^t.dt - w_n\int_0^1 tv^t.dt) + \frac{w_n v^{n-1}}{l_x}\int_0^1 v^t.dt$$

$$= \frac{\delta_n v^{n-1}}{l_x.l_{x+n-1}}\left\{\frac{v-1}{\log v}\lambda_{n-1} + \frac{1}{\log v}\left(\frac{v-1}{\log v} - v\right)w_n\right\} + \frac{v-1}{\log v}\cdot\frac{w_n v^{n-1}}{l_x}.^*$$

Hence, summing with respect to n (from 1 to n) we get the whole value of the risk of death or withdrawal during the n years,

$$= \frac{v-1}{l_x.\log v}\left(\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} + \Sigma w_n v^{n-1}\right) + \frac{1}{l_x.\log v}\left(\frac{v-1}{\log v} - v\right)\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} \quad ..(2).$$

From (1) we readily find that

$$\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} + \Sigma w_n v^{n-1} = \tfrac{1}{2}\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} + \Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n \quad ...(3);$$

hence (2) becomes, by substitution and reduction,

$$\frac{1}{l_x.\log v}\left\{\frac{v-1}{\log v} - \tfrac{1}{2}(v+1)\right\}\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} + \frac{v-1}{l_x.\log v}(\Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n) \quad ..(4),$$

a formula which involves only two different summations.

This, multiplied by P_x, gives a complete and perfectly general value of the risk of death and withdrawal.

If we suppose the probability of withdrawal in any year to be always the same at the commencement of the year and $=k$, then $\frac{w_n}{\lambda_{n-1}} = k$, or $w_n = k\lambda_{n-1}$; and substituting in (3), we get

$$(1-\tfrac{1}{2}k)\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} = (1-k)\Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n \quad ...(5);$$

also the quantity $\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}}$, in (4), becomes $k\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}}$; which, by (5), is

$$= \frac{k(1-k)}{1-\tfrac{1}{2}k}\Sigma\lambda_{n-1}v^{n-1} - \frac{k}{v(1-\tfrac{1}{2}k)}\Sigma\lambda_n v^n;$$

and this substituted in (4), gives, if we put β for $\frac{1}{l_x.\log v}\left(\frac{v-1}{\log v} - \tfrac{1}{2}(v+1)\right)$,

$$\left(\frac{\beta k(1-k)}{1-\tfrac{1}{2}k} + \frac{v-1}{l_x.\log v}\right)\Sigma\lambda_{n-1}v^{n-1} - \left(\frac{\beta k}{v(1-\tfrac{1}{2}k)} + \frac{v-1}{v.l_x.\log v}\right)\Sigma\lambda_n v^n \quad ..(6)$$

for the value of the risk of death or withdrawal on the particular supposition named. Here, only the one summation, $\Sigma\lambda_n v^n$, is required to be made. When this formula is used, the values of λ_1, λ_2, λ_3, &c., are computed from the simple relation

$$\lambda_n = \lambda_{n-1}\{(1-\tfrac{1}{2}k)p_{x+n-1(1)} - \tfrac{1}{2}k\}.$$

If we find Q to be the arithmetical value of either (4) or (6), then we must have

$$Q.P_x + \frac{N_{x+n}}{D_x} = P_x, \text{ or } P_x = \frac{N_{x+n}}{(1-Q)D_x} \quad .....(7).$$

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* The general integrals are $\int v^t.dt = \frac{v^t}{\log v}$, and $\int tv^t dt = \frac{tv^t}{\log v} - \frac{1}{\log v}\int v^t dt$ $= \frac{v^t}{\log v}\left\{t - \frac{1}{\log v}\right\}$.

By way of illustration, let 50 be the age at entry, and 60 the age at which the annuity (£1) is to be entered upon, and suppose it has been found that the premiums on one-twentieth of the policies in force at the beginning of any year are withdrawn during that year; then, using the Carlisle table of mortality, and 4 per cent. interest, we shall find the value of the expression (6) to be $\cdot 42556 = Q$; hence, from (7) we get $P_x = 9\cdot 4157$.

In the same case, if the premium were simply returnable at the end of the year of death, and no option of withdrawal were allowed, the single premium, by the ordinary formula, would be

$$P_x = \frac{N_{x+n}}{D_x - M_x + M_{x+n}} = 6\cdot 2631.$$

These numerical results show that a very moderate supposition as to the probability of withdrawal increases the premium more than 50 per cent.

It will be observed, in all that precedes, that $\log v$ denotes the *hyperbolic* logarithm of v.

I am, Sir,
Your obedient servant,
SAMUEL YOUNGER.

316, Regent Street, London,
30th November, 1865.

THE LATE MR. FINLAISON'S TABLES.

To the Editor.

SIR,—Reference having been made at the last meeting of the Institute to the discrepancy existing between the probabilities of living as computed by the late Mr. Finlaison, and those that would result from the methods of graduation he professes to adopt, I trouble you with a few details of these differences, with a view of promoting some inquiry into a matter which, having regard to the importance of the tables, and as bearing upon the points recently under discussion, may prove of interest to members.

The well-known tables of Mr. Finlaison are embodied in his Report, printed by order of the House of Commons, 31st March, 1829; and are founded on twenty-one "Observations" exhibiting the rate of mortality experienced under the Government schemes of tontines and other annuities, with an annuity table at 4 per cent. deduced from each. The logarithms of the probability of living one year at each age are tabulated, first as deduced from the data, and secondly as adjusted by Mr. Finlaison.

His formulæ for adjustment, written in Milne's notation, are embodied in the Report, and are as follows:—

$$\text{I.}\quad {}_1\bar{\bar{a}} = \sqrt[3]{\left\{\begin{array}{l} \sqrt[5]{{}_1a_3 \times {}_1a_2 \times {}_1a_1 \times {}_1a \times {}^1_1a}\} \times \\ \sqrt[5]{{}_1a_2 \times {}_1a_1 \times {}_1a \times {}^1_1a \times {}^2_1a}\} \times \\ \sqrt[5]{{}_1a_1 \times {}_1a \times {}^1_1a \times {}^2_1a \times {}^3_1a}\} \end{array}\right\}}.$$

$$\text{II.}\quad {}_1\bar{\bar{a}} = \frac{(5\,{}_1a + 4\,{}_1a_1 + 4\,{}^1_1a + 3\,{}_1a_1 + 3\,{}^2_1a + 2\,{}_1a_3 + 2\,{}^3_1a + {}_1a_4 + {}^4_1a)}{25}.$$

Mr. Finlaison informs us that nineteen of the observations were "all and each" of them adjusted by the first method above; and that two observations only, the earliest for each sex, which were completed in January, 1823, were adjusted by the second method, which he designates as "perhaps quite as good, but more laborious" than the first; and that by these last mentioned "the whole of the life annuities for the service of Government have been calculated." The two observations specified appear, by the repetition of the above notice in their titles, to be the 13th and 20th, being on female and male lives respectively. I shall take my examples from the latter, as in addition to its other uses it is that by which the succession duties are levied under Act 16th and 17th Vict., cap 51.

The logarithms of the unadjusted probabilities, as given by Mr. Finlaison, require correction. In two instances they can only result by the alteration of a figure in the number living; in these cases I have retained the logarithm given, assuming that the number entering on age 12 should be 4,904, and that at age 73 should be 1,323. At age 16, however, the alteration of one figure is demanded in the logarithm, which I have therefore read ·9977864, instead of ·9975864.

With these explanations, I now annex the probability of living one year at the ages indicated—as deduced from the exact mortality, as corrected by Mr. Finlaison, and as resulting from both his methods of adjustment above quoted; together with the probability that would be shown by treating the logarithms as numbers in the second formula, the alternative having suggested itself of that being the cause of the discrepancy.

The difference of each from Mr. Finlaison's is sufficiently great to require explanation.

Age.	Probability, from Data, that Life survive one Year.	Probability as adjusted by Mr. Finlaison.	Probability resulting from Mr. Finlaison's second Method of Adjustment.	Probability resulting from first Method.	Probability resulting from treating the Logs. by second Method.	Age.
15	·9951015	·9940560	·9942626	·9945517	·9942615	15
25	·9865307	·9860189	·9862160	·9859922	·9862152	25
35	·9895447	·9875697	·9878904	·9880562	·9878895	35
45	·9876687	·9863270	·9868670	·9869892	·9868667	45
55	·9768117	·9749987	·9760568	·9760392	·9760505	55
65	·9632754	·9592738	·9607349	·9616342	·9607157	65

You will not fail to notice that, with one trifling exception, the results *correctly* deduced by Mr. Finlaison's formulæ exhibit a greater probability of living than those given by himself; and it is a subject for legitimate speculation whether some unexplained reason, not appearing in his Reports, may not account for the great difference between his annuity values of the two sexes.

I may add that I have proved the correct application of the first method to the results as detailed in the 8th observation.

A few more discrepancies in the (20th) table under consideration are worthy of notice. The fraction measuring the probability of death is, in several cases, inconsistent with the number resulting from the published logarithm of the corrected probability of living. Thus, at age 15, for ·0059440, read ·0059493; at age 41, for ·0134731, read ·0133368; at age 71, for ·0656881, read ·0656860.

I am, Sir,
Your most obedient servant,
H. AMBROSE SMITH.

7, Torrington Square, W.C.,
22nd December, 1865.

THE COMPARATIVE MERITS OF THE OLD AND NEW METHODS OF SOLUTION.

To the Editor.

Sir,—I select the enclosed from some old papers on which I have just put my hands. It was not prepared with any view to publication, and I send it you without the change of a word.

It appears to me deserving of notice from its bearing on the question of the comparative merits of the old and the new methods of solution—Barrett's and Milne's. The facility with which the result is obtained, and the *aptitude* (so to speak) with which the various elements combine to form an expression that so plainly tells its own tale, are very striking and instructive.

I am, Sir,
Yours obediently,

London, 17*th November*, 1865. P. GRAY.

Problem.

A man, (x), and his wife (y), wish to provide an annuity of £a, to be enjoyed by (y) after the death of (x), if within n years; but to be entered upon in n years if both or either be then alive, and to continue till both lives are extinct. Required the annual premium, ϖ, payable till the annuity is entered upon.

The annuity consists of two distinct portions, the first during, and the second after, n years. The first is a temporary reversionary annuity on (y) after (x); and its present value, therefore, is $a\left(\frac{N_{y|n}}{D_y}-\frac{N_{x.y|n}}{D_{x.y}}\right)$. The second is a joint life and survivor annuity deferred n years, the present value of which is $a\left(\frac{N_{x+n}}{D_x}+\frac{N_{y+n}}{D_y}-\frac{N_{x+n.y+n}}{D_{x.y}}\right)$. The present value of the benefit therefore is

$$a\left(\frac{N_{y|n}}{D_y}-\frac{N_{x.y|n}}{D_{x.y}}+\frac{N_{x+n}}{D_x}+\frac{N_{y+n}}{D_y}-\frac{N_{x+n.y+n}}{D_{x.y}}\right)=a\left(\frac{N_y}{D_y}+\frac{N_{x+n}}{D_x}-\frac{N_{x.y}}{D_{x.y}}\right).$$

And since the premium will cease on the occurrence of either of the two events, the death of x or the lapse of n years, whichever should happen first, its present value will be $\frac{\varpi N_{x-1|n}}{D_x}$.

Equating this with the foregoing expression, we find

$$\varpi=\frac{aD_x\left(\frac{N_y}{D_y}+\frac{N_{x+n}}{D_x}-\frac{N_{x.y}}{D_{xy}}\right)}{N_{x-1|n}}.$$

Scholium.—From an inspection of the expression for the present value of the benefit, it appears that, agreeably to the method called by Professor De Morgan "The balance of annuities," the conditions of the benefit will be fulfilled by granting an annuity to both (y) and (x); that of the former to commence now, and that of the latter in n years; and to withdraw one of them so long as both (y) and (x) survive. And a little consideration shows that this is just.

P. G.

London, 21*st September*, 1844.

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On the Construction of Tables by the Method of Differences.

By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

[Read before the Institute, 26th March, 1866.]

INTRODUCTION.

EVERY mathematical table consists of a series of values of a function corresponding to successive values of the variable, which last series of values forms the *argument* of the table. Any such table may be constructed therefore, when the function to be tabulated—which may be called the *characteristic function*—is known, by the evaluation of that function in terms of the successive values of the argument. It is only, however, when the table to be constructed is of limited extent that this method of formation would be employed. If the table be extensive, and especially if the characteristic function be complex, this, which may be called the *direct* method, would become too laborious, each value when formed in this way also requiring separate verification. In these circumstances the Method of Differences becomes available for the end in view. This method dispenses with all reference to the characteristic function beyond what is necessary for the formation of a few values (which I call *fundamental* or *primitive values*), at stated intervals; and in applying it, each value being dependent on the preceding, verification is obtained by the periodical coincidence with those fundamental values of the corresponding terms in the series in course of being formed.

It is this method which I propose, in the following pages, to investigate and exemplify. It is an elementary application of the Calculus of Finite Differences, in the systematic treatises on which Calculus, however, it receives almost no attention. Mr. Woolhouse has contributed to the pages of this *Journal** a series of papers on "Interpolation, Summation, and the Adjustment of Numerical Tables," which are characterized by that gentleman's usual elegance and ability; but I think it will be found that his object in the papers referred to and that which I have here in view in no respect interfere with each other.

For the sake of perspicuity the matter of the following essay will be divided into four sections. In the first I shall explain and exemplify Horner's process for the transformation of functions. This process, which is one of great utility, is not nearly so well known as it deserves to be; and I hope the space devoted to its elucidation will not be considered wasted. In the second section the theorems of the Calculus of Finite Differences for which we have occasion will be briefly established. The third section will be devoted to the construction of tables in which the characteristic function is rational and integral. And the fourth section, to which the third will be only introductory, will be occupied with the construction of tables in which the characteristic function is irrational or transcendental. The two sections last named, especially the fourth, will comprise copious examples of the application of the principles established in the previous sections to the actual construction of tables.

SECTION I.—*On Horner's Method of Transformation.*

Problem I.

(1.) To divide, synthetically, $\phi(x)$, a given rational and integer polynomial of n dimensions in x, by the binomial $x-a$, where a is any number, positive or negative.

If the polynomial to be divided be

$$Ax^n+Bx^{n-1}+\ldots+Lx^2+Mx+N,$$

we shall have for quotient,

$$A_1x^{n-1}+B_1x^{n-2}+\ldots+K_1x^2+L_1x+M_1,$$

a polynomial of $n-1$ dimensions, and in which the coefficients are as yet undetermined; and for remainder, R_1. Hence—

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* Vol. xi., pp. 61, 301; and vol. xii., p. 136.

$$Ax^n+Bx^{n-1}+\ldots+Lx^2+Mx+N=$$
$$(A_1x^{n-1}+B_1x^{n-2}+\ldots+K_1x^2+L_1x+M_1)(x-a)+R_1.$$

Performing the multiplication here indicated, the second member of this equation becomes

$$A_1x^n+(B_1-A_1a)x^{n-1}+(C_1-B_1a)x^{n-2}+\ldots+(L_1-K_1a)x^2+$$
$$(M_1-L_1a)x+R_1-M_1a.$$

Hence, equating coefficients, the two members being identical,

$$\begin{array}{lcl} & A_1=A & \\ B_1-A_1a=B & & \therefore B_1=A_1a+B \\ C_1-B_1a=C & & C_1=B_1a+C \\ D_1-C_1a=D & & D_1=C_1a+D \\ \vdots & & \vdots \\ L_1-K_1a=L & & L_1=K_1a+L \\ M_1-L_1a=M & & M_1=L_1a+M \\ R_1-M_1a=N & & R_1=M_1a+N \end{array}$$

(2.) It thus appears that the first coefficient of the quotient is equal to the first of the dividend; and that the others, including the remainder, are derived, each from that which precedes it, by a simple and uniform operation. This operation consists in multiplying each coefficient as it arises by a, and adding the product to the next coefficient of the dividend.

(3.) The process may be typified as follows :—

$$\begin{array}{llllll} A & +B & +C\ldots\ldots+L & +M & +N\,(a \\ & +Aa & +B_1a \quad +K_1a & +L_1a & +M_1a \\ \hline A & +B_1 & +C_1\ldots+L_1 & +M_1 & +R_1 \end{array}$$

where the coefficients A, B_1, C_1, &c., are respectively the sums of the quantities standing over them.

(4.) The following is a numerical example :—

Divide $2x^3-25x^2-4x+50$ by $x-10$.

$$\begin{array}{rrrl} 2 & -25 & -4 & +50\,(10=a \\ & 20 & -50 & -540 \\ \hline 2 & -5 & -54 & -490 \end{array}$$

The quotient is $2x^2-5x-54$, and the remainder is -490.

(5.) If a be essentially negative, by using $-a$ as the multiplier we obtain the quotient arising from division by $x+a$. Thus :—

$$\begin{array}{rrrl} 2 & -25 & -4 & +50\,(-10 \\ & -20 & +450 & -4460 \\ \hline 2 & -45 & +446 & -4410 \end{array}$$

The quotient here is $2x^2-45x+446$, and the remainder is -4410.

(6.) When the coefficients are not large numbers, and especially when the multiplier does not exceed 10, the foregoing operation may be very conveniently conducted without setting down the addends. It is recommended to practise this method, in accordance with which the example of (4) will assume the following form:—

$$\begin{array}{rrrr} 2 & -25 & -4 & +50\,(10 \\ & -5 & -54 & -490 \end{array}$$

(7.) In applying the foregoing process, in which the coefficients are detached from their arguments, to an *incomplete* polynomial—that is, a polynomial in which some of the powers of x are wanting—it is obvious that, to avoid error, it must be rendered *complete*, by indicating the places of the missing powers. This is done by inserting those powers, with zero coefficients. Thus, for the present purpose, $3x^4-2x^2+6x$ would require to be written $3x^4+0x^3-2x^2+6x+0$; and to divide this polynomial by $x-4$, we should proceed as follows:—

$$\begin{array}{rrrrr} 3 & 0 & -2 & 6 & 0\,(4 \\ & 12 & 46 & 190 & 760 \end{array}$$

The result is, quotient $3x^3+12x^2+46x+190$, and remainder 760.

(8.) Denoting the polynomial to be divided by $\phi(x)$, the quotient by Q, and the remainder by R, we obviously have

$$\phi(x)=\mathrm{Q}(x-a)+\mathrm{R}.$$

And if we write here a for x, we get

$$\phi(a)=\mathrm{R}.$$

That is, the remainder arising from the division of $\phi(x)$ by $x-a$, is what $\phi(x)$ becomes by the substitution in it of a for x. Thus, in the examples of (4) and (5), -490 and -4410 are what the polynomial operated upon becomes when 10 and -10 are respectively substituted in it for x.

(9.) From the foregoing it appears, that besides an easy method of performing division by $x-a$, we have here also a compact and commodious process for substituting in any polynomial a specific value for the variable.

(10.) The theorem just proved is one of some interest and importance, and we shall have to refer to it hereafter. I therefore give another demonstration of it. Let the polynomial be

$$\mathrm{A}x^3+\mathrm{B}x^2+\mathrm{C}x+\mathrm{D}.$$

Then, operating upon it as directed, the coefficients of the quotient will be—

$$\begin{array}{ll}\text{the first,} & \mathrm{A};\\ \text{second,} & \mathrm{A}a+\mathrm{B};\\ \text{third,} & (\mathrm{A}a+\mathrm{B})a+\mathrm{C}=\mathrm{A}a^2+\mathrm{B}a+\mathrm{C};\end{array}$$

and the remainder will be

$$(\mathrm{A}a^2+\mathrm{B}a+\mathrm{C})a+\mathrm{D}=\mathrm{A}a^3+\mathrm{B}a^2+\mathrm{C}a+\mathrm{D},$$

which is the given polynomial, with a substituted in it for x.*

Problem II.

(11.) To transform $\phi(x)$, a rational and integer function of x, of n dimensions, into $\phi(x+a)$; in other words, having given $\phi(x)=\mathrm{A}x^n+\mathrm{B}x^{n-1}+\mathrm{C}x^{n-2}+\ldots+\mathrm{M}x+\mathrm{N}$, to expand $\phi(x+a)$ in powers of x.

Divide $\phi(x)$ by $x-a$, and let the quotient and the remainder be Q_1 and R_1, respectively; divide Q_1 by $x-a$, and let the quotient and the remainder of this division be Q_2 and R_2 respectively. Proceed in this way, dividing each quotient as it arises by $x-a$, till Q_n and R_n shall have been formed. The required expansion will then be

$$\phi(x+a)=\mathrm{A}x^n+\mathrm{R}_nx^{n-1}+\mathrm{R}_{n-1}x^{n-2}+\ldots.+\mathrm{R}_2x+\mathrm{R}_1.$$

The rule may be exemplified thus:—

$$\begin{array}{rcc} x-a) & \phi(x) & \\ \hline x-a) & \mathrm{Q}_1 & \mathrm{R}_1 \\ \hline x-a) & \mathrm{Q}_2 & \mathrm{R}_2 \\ & \vdots & \vdots \\ \hline & \mathrm{Q}_n & \mathrm{R}_n \end{array}$$

(12.) From the foregoing operation we have

$$\begin{aligned}\phi(x)&=\mathrm{Q}_1(x-a)+\mathrm{R}_1;\\ \mathrm{Q}_1&=\mathrm{Q}_2(x-a)+\mathrm{R}_2;\\ &\;\vdots\\ \mathrm{Q}_{n-1}&=\mathrm{Q}_n(x-a)+\mathrm{R}_n.\end{aligned}$$

Hence, by successive substitution,

$$\begin{aligned}\phi(x)&=\mathrm{Q}_1(x-a)\mathrm{R}_1=\mathrm{Q}_2(x-a)^2+\mathrm{R}_2(x-a)+\mathrm{R}_1\\ &=\mathrm{Q}_3(x-a)^3+\mathrm{R}_3(x-a)^2+\mathrm{R}_2(x-a)+\mathrm{R}_1\\ &\;\vdots\\ &=\mathrm{Q}_n(x-a)^n+\mathrm{R}_n(x-a)^{n-1}+\ldots.+\mathrm{R}_2(x-a)+\mathrm{R}_1.\end{aligned}$$

---

* The process of this problem receives the name of Synthetic Division; and it is easily extended to the case in which the divisor is a polynomial of the form $x^m+ax^{m-1}+bx^{m-2}\ldots$ See Hutton's *Course*, by Davies, vol. ii., pp. 127 to 130, and 524; also *Penny Cyclopædia*, Supplement, article "Power."

Writing now in this equation, on both sides, $x+a$ for x, we obtain

$$\phi(x+a)=Q_n x^n+R_n x^{n-1}+R_{n-1}x^{n-2}+\ldots\ldots+R_2x+R_1.$$

Now, $Q_n=A$. For, owing to the form of the divisor, A, the leading coefficient of $\phi(x)$, is also the leading coefficient of each of the quotients. And Q_n being obviously of no dimensions with respect to x, it is a constant, and therefore equal to A. The expansion consequently is, finally,

$$\phi(x+a)=Ax^n+R_n x^{n-1}+\ldots\ldots+R_2x+R_1.$$

(13.) In applying this theorem to the transformation of functions, the divisions will be effected by the method of Problem I., in consequence of which the operation assumes a very commodious and compact form. The following is a type of it, in which, as before, the quantities beneath the lines are respectively the algebraical sums of the two above them.

Let $\phi(x)$ be of the fourth degree,

$$Ax^4+Bx^3+Cx^2+Dx+E.$$

$$\begin{array}{lllll}
{}^{*}A & B & C & D & E\,(a \\
 & Aa & B_1a & C_1a & D_1a \\
\hline
A & B_1 & C_1 & D_1 & R_1 \\
 & Aa & B_2a & C_2a & \\
\cline{1-4}
A & B_2 & C_2 & R_2 & \\
 & Aa & B_3a & & \\
\cline{1-3}
A & B_3 & R_3 & & \\
 & Aa & & & \\
\cline{1-2}
A & R_4 & & &
\end{array}$$

Here

$$\begin{aligned}
Q_1&=Ax^3+B_1x^2+C_1x+D_1,\\
Q_2&=Ax^2+B_2x+C_2,\\
Q_3&=Ax+B_3,\\
Q_4&=A;
\end{aligned}$$

and, the remainders being as indicated, the expansion is

$$\phi(x+a)=Ax^4+R_4x^3+R_3x^2+R_2x+R_1.$$

(14.) Comparing this with the expansion of the same function by Taylor's theorem, viz.,

$$\phi(x+a)=\phi(a)+\phi_1(a)x+\frac{\phi_2(a)}{1.2}x^2+\frac{\phi_3(a)}{1.2.3}x^3+\frac{\phi_4(a)}{1.2.3.4}x^4,$$

where $\phi_n(a)$ is the nth differential coefficient of $\phi(x)$ when a is put in it for x, we see that

---

* The first coefficient remaining constant throughout the operation, it is unnecessary in practice to repeat it on the successive lines.

$$R_1=\phi(a),\ R_2=\phi_1(a),\ R_3=\frac{\phi_2(a)}{1.2},\ R_4=\frac{\phi_3(a)}{1.2.3},\ \text{and } A=\frac{\phi_4(a)}{1.2.3.4}.$$

For none of these, however, except the first, shall we have any occasion; and it has been established already (8, 10).

(15.) For a numerical example, transform $\phi(x)=2x^3-25x^2-4x+50$, into $\phi(x+10)$.

$$\begin{array}{rrrrl}
2 & -25 & -4 & +50 & (10 \\
 & -5 & -54 & -490 & \\
 & 15 & 96 & & \\
 & 35 & & &
\end{array}$$

$$\therefore\ \phi(x+10)=2x^3+35x^2+96x-490.$$

(16.) Again, transform $\phi(x)$ into $\phi(x-10)$.

$$\begin{array}{rrrrl}
2 & -25 & -4 & +50 & (-10 \\
 & -45 & 446 & -4410 & \\
 & -65 & 1096 & & \\
 & -85 & & &
\end{array}$$

$$\therefore\ \phi(x-10)=2x^3-85x^2+1096x-4410.$$

(17.) Looking to what is accomplished by the operation now exemplified, it is clear that we ought to have the same result whether we transform with a or with any numbers whose sum is a. For example, to transform $\phi(x)$ into $\phi(x+2)$, $\phi(x+2)$ into $\phi(\overline{x+2}+3)$ or $\phi(x+5)$, and $\phi(x+5)$ into $\phi(\overline{x+5}+5)$ or $\phi(x+10)$, ought to be the same thing as to transform, by a single operation, $\phi(x)$ into $\phi(x+10)$. That this is true the following shows:—

$$\begin{array}{lrrrrl}
\phi(x) & 2 & -25 & -4 & +50 & (2 \\
 & & -21 & -46 & -42 & \\
 & & -17 & -80 & & \\
 & & -13 & & & \\
\cline{2-5}
\phi(x+2) & 2 & -13 & -80 & -42 & (3 \\
 & & -7 & -101 & -345 & \\
 & & -1 & -104 & & \\
 & & 5 & & & \\
\cline{2-5}
\phi(x+5) & 2 & 5 & -104 & -345 & (5 \\
 & & 15 & -29 & -490 & \\
 & & 25 & 96 & & \\
 & & 35 & & & \\
\cline{2-5}
\phi(x+10) & 2 & 35 & 96 & -490, & \text{as before (15).}
\end{array}$$

(18.) It will add to the interest connected with the process now elucidated if I diverge for a little from the object more immediately in view, to call attention to a very remarkable application of the process in question.

(19.) It has been already more than once shown (8, 10, 14), that the last term—the *absolute* term—of $\phi(x+a)$ is $\phi(a)$, the result of the substitution of a for x in $\phi(x)$. The same thing appears also if we make x in $\phi(x+a)$ equal to 0; for we then have simply $\phi(a)$. From this it follows, that if a be a root* of $\phi(x)$, the absolute term of $\phi(x+a)$ will be equal to nothing. Hence also, conversely, if the absolute term of $\phi(x+a)$ be nothing, it will follow that a, the number, or the sum of the numbers, used in the transformation, is a root of $\phi(x)$.

(20.) An instance will illustrate this. Let, as before,

$$\phi(x)=2x^3-25x^2-4x+50;$$

and transform it into $\phi(x+12\cdot5)$, thus—

	2	−25	−4	50 (12·5
		0	−4	0
		25	308·5	
		50		
$\phi(x+12\cdot5)$	2	50	308·5	0

or thus—

	2	−25	−4	50 (10	$50=\phi(0)$
		−5	−54	−490	$-490=\phi(10)$
		15	96		
		35			
$\phi(x+10)$	2	35	96	−490 (2	
		39	174	−142	$-142=\phi(12)$
		43	260		
		47			
$\phi(x+12)$	2	47	260	−142 (·5	
		48	284	0	$0=\phi(12\cdot5)$
		49	308·5		
		50			
$\phi(x+12\cdot5)$	2	50	308·5	0	

We thus find $\phi(x+12\cdot5)=2x^3+50x^2+308\cdot5x$, in which the absolute term is nothing; and 12·5 is a root of $\phi(x)$, as will be found on trial.

(21.) The transforming number a being necessarily rational, it will appear at once that we can fulfil the condition $\phi(a)=0$, or, in other words, cause the absolute term of $\phi(x+a)$ to vanish, only when $\phi(x)$ has a rational root. When the root or roots of $\phi(x)$

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* By a root of a polynomial, as $\phi(x)$, is meant a number which, substituted in it for x, causes it to vanish. It is the same thing, therefore, as a root of the equation $\phi(x)=0$.

are irrational, however, these roots can always, as in other such cases, be represented by rational numbers, which shall fulfil the condition $\phi(x)=0$ as nearly as we please.

(22.) The result just established may be also deduced in a somewhat different manner. The substitution of any quantity r for x in $\phi(x+a)$ gives the same result as the substitution of $r+a$ for x in $\phi(x)$: each function then becomes $\phi(r+a)$. Hence, if r be a root of $\phi(x+a)$, $r+a$ will be a root of $\phi(x)$. If now r, still a root of $\phi(x+a)$, be equal to nothing, $0+a$, or a, will be a root of $\phi(x)$. But nothing cannot be a root of $\phi(x+a)$ unless the absolute term of this function be equal to nothing, as it is only terms having x as a factor that will vanish for $x=0$. We are hence conducted to the same conclusion as before, namely, that when $\phi(x+a)$ has nothing for its absolute term, a is a root of $\phi(x)$.

(23.) From what precedes, then, it appears that if we can, by a series of transformations upon any given function, $\phi(x)$, succeed in producing a transformed function, $\phi(x+a)$, whose absolute term is nothing, we shall have determined a root of $\phi(x)$; that is, we shall have obtained a solution of the equation $\phi(x)=0$, the root being a, the number, or the sum of the numbers, used in the transformation. And we shall have an additional root of $\phi(x)$ for every different value of a, which will enable us to bring $\phi(x+a)$ into the required form.

(24.) We should thus seem to be put in possession of a general method for the solution of numerical equations of all orders; and, with a little extraneous aid, such is the case. Before we can apply the transforming process with success, we require to know the *nature* and the *situation* of the roots of the equation to be solved; and this information we procure by the aid of considerations supplied in the theory of equations.

(25.) I now give an example of the solution of a cubic.

Required the roots of $2x^3-25x^2-4x+50$; that is, it is required to solve the equation,

$$2x^3-25x^2-4x+50=0.$$

Analyzing this equation, we learn, first, that the three roots are all real; secondly, that two of them are positive and one negative; and, thirdly, that the positive roots are comprised in the intervals [1, 2] and [10, 20] respectively; and the negative root in the interval [-2, -1]. I shall first develop the positive root in the interval [1, 2].

$\phi(x+0)$	2	−25	−4	50 (1·	50=$\phi(0)$
		−23	−27	23	23=$\phi(1)$
		−21	−48		
		−19			
$\phi(x+1)$	2	−19	−48	23(·4	
		−18·2	−55·28	·888	·888=$\phi(1·4)$
		−17·4	−62·24		
		−16·6			
$\phi(x+1·4)$	2	−16·6	−62·24	·888(·01	
		−16·58	−62·4058	·263942	·263942=$\phi(1·41)$
		−16·56	−62·5714		
		−16·54			
$\phi(x+1·41)$	2	−16·54	−62·5714	·263942	

(26.) The first three transformations, conducted in accordance with previous examples, are shown above. The object aimed at is to reduce the absolute term, 50, as nearly to zero as we can. The first transformation, by 1, reduces it to 23, $=\phi(1)$; the second, by ·4, to ·888, $=\phi(1·4)$; and the third, by ·01, to ·263942, $=\phi(1·41)$. The first transforming number, 1, is known, being the less of the limits comprising the root; the next, ·4, is the quotient of 23, the last coefficient, by 48, the preceding (anticipating, as well as we can, the effect of the next transformation); the third, ·01 is obtained in the same way, but with greater certainty; and, if we were to discontinue our transformations here, the work has now become so convergent that we should obtain the next three figures of the root true by the simple division of ·2639 . . . by 62·57 . . .

(27.) But it would be tedious to proceed further as above. The working can be much abbreviated and simplified by the aid of the following proposition in the theory of equations:—

If the terms of a polynomial (made complete (7), if necessary) be multiplied in order by the same number of successive terms of the series . . . m^{-3}, m^{-2}, m^{-1}, 1, m, m^2, m^3 . . . , the roots of the polynomial so transformed will be respectively those of the original polynomial multiplied by m.

(28.) Thus, the roots of x^3-7x+6 are 1, 2, and −3. And if, writing it x^3+0x^2-7x+6, we multiply the several terms by 10, 10^2, 10^3, 10^4, we get $10x^3-7000x+60000$, the roots of which will be found to be 10, 20, and −30.

(29.) It is usual, in employing this transformation, in order that the coefficients be not needlessly increased, to commence with unity as the first multiplier; so that the given polynomial would have been with more propriety transformed into $x^3-700x+6000$, the roots of which are still, 10, 20, and -30.*

(30.) I now résume the equation in (25). The following is the complete solution, as far as twelve places:—

```
2   —25                 —4                      50(1·414,21,356237
    —23                 —27       ..........23000
    —21       .......—4800..: ..........888000
    —190...:            —5528     : ........263942000
    —182      .......—622400..: : .......13391888
    —174      :         —624058   : : .......850502
    —1660..:     ....—62571400..: : : .....223398
    —1658       :       —62637528 : : :     35266
    —1656     ..: ..—62703624̸....: : :      3911
    —16540..:     : —6270693      - : :       148
    —16532        : —6271023̸.......: :        23
    —16524     ...: —627104           :         4
    —16̸51̸6...:      —627106̸..........:         0
```

$$x=1{\cdot}41421356237$$

(31.) The first modification here is, that the formality of re-writing the coefficients of the several transformed polynomials is dispensed with, so that they no longer range in line with each other. I have indicated their connexion, however, by rows of dots. But this is not necessary, as will be seen.

(32.) A second modification is, that, by aid of the theorem in (27), the roots of each transformed polynomial as it arises are multiplied by 10; the effect of which is that each root figure is brought in succession into the unit's place, so far at least as to admit of its being employed in the next transformation as if belonging to that place. The ciphers annexed to the several coefficients for the purpose of this modification suffice to point out the completion of each transformation.

(33.) The third modification is but an adaptation of the second. When the last coefficient has been sufficiently extended to afford as many root figures as we want, the orderly multiplication of the roots is thenceforward effected by cutting off one figure from the first preceding coefficient, two from the second preceding, and so on. At the first curtailment, as here, the first coefficient usually disappears, and the operation becomes that for the solution of an equation of the next lower order—a quadratic. At the second the second coefficient here *nearly* disappears, and the operation ultimately merges into contracted division, which gives the last six of the root figures. I have marked by commas in the root the points where the operations change.

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* The demonstration of the above theorem, which is very simple, will be found in any treatise on the theory of equations. The most accessible of these is probably Professor J. R. Young's *Analysis and Solution of Cubic and Biquadratic Equations.* This work, although strictly elementary, goes really a long way into the subject.

(34.) It is apparent now, in this case at least—and it is also true *generally**—that it suffices to know the first figure of a root to enable us to evolve that root, figure by figure, either exactly, if rational, or, if irrational, to any degree of approximation that may be desired. I will illustrate this point. The second figure, 4, is suggested by the mental division of 23000 by 4800; and 4 succeeds, since the absolute term is reduced to 888, two places lower in the scale, and does not change sign, while it is easy to see that 5 would be too great. The next figure, 1, is suggested, with almost certainty, by division of 888... by 622..., and it succeeds. In fact this division would give the two figures, 1, 4. The next trial division, 2639... by 625... would give the three figures, 4, 2, 1. And thus as we advance the work becomes more and more convergent till, the first and second columns having disappeared, and so lost all power of influencing the third, the number at the bottom of this, 627106, being used with that *then* at the bottom of the last, namely, 223398, as divisor and dividend, the quotient is the last six figures of the root, namely 356237.

(35.) It is much less easy to *follow* a written example in an operation of this kind than to work it for oneself. I therefore suggest the working of the example to any one who wishes to master the process, and who finds difficulty in following the description.

(36.) I have finally here to point out that the absolute term has, as the result of the operation, been diminished till it has no significant figure in the first nine decimal places.

(37.) I shall now develop the negative root which we found to subsist in the interval $[-2, -1]$, and whose first figure is therefore -1, in the unit's place.

(38.) It is somewhat inconvenient to use a negative transforming number, and, therefore, when a negative root is to be evolved, it is customary, as a preliminary, to convert it into a positive root. This is done by the aid of the following theorem.

(39.) If the signs of the alternate terms of a polynomial (rendered complete) be changed, the signs of all the roots of that polynomial will be changed.

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* It is not true *universally*, however. In rare cases two or more roots may be situated so closely together that a somewhat perplexing preliminary analysis is requisite to effect their complete separation. But the separation can *always* be made.

(40.) Thus, the roots of x^3-7x+6 being 1, 2, and -3, we shall have for the polynomial whose roots are -1, -2, and 3,

$$\text{either} \quad x^3-7x-6,$$
$$\text{or} \quad -x^3+7x+6;$$

as the change may be made in either the first, third, &c., or the second, fourth, &c., terms.

(41.) So also we change the signs of the roots of the polynomial $2x^3-25x^2-4x+50$, by writing it $2x^3+25x^2-4x-50$; and its negative root, now become positive, is developed as follows:—

2	25	−4	−50(1·414,21,356238
	27	23	−27000
	29	5200	−1112000
	310	6472	−331058000
	318	777600	−16808112
	326	780942	−1067497
	3340	78428600	−280396
	3342	78562472	−44265
	3344	7869637$\cancel{6}$	−4910
	33460	7870307	−187
	33468	787097$\cancel{7}$	−30
	33476	787101	−6
	3$\cancel{3}$4$\cancel{8}$4	78710$\cancel{4}$	0

(42.) The analogy of this development to that in (30) supersedes the necessity for any general remarks here. The identity, except as to sign, of the two roots will, of course, be observed. This would hardly arise except in a polynomial specially framed, as the one whose roots are being developed has been. It is the product $(2x-25)(x^2-2)$, or $2(x-12{\cdot}5)(x-\sqrt{2})(x+\sqrt{2})$, of which the three roots obviously are

$$12{\cdot}5, \quad \sqrt{2}, \quad \text{and} \quad -\sqrt{2}.$$

It is the last two roots, $\pm\sqrt{2}$, that have been developed in (30) and (41). The last figure is truly 7, as in (30). The roots could have easily been carried to double the number of decimal places.

(43.) The rational root, 12·5 was developed in (20). I repeat the development here, for the purpose of introducing a modified application of the theorem in (27). By multiplying the coefficients of a polynomial by successive terms of the series $\ldots m^{-2}, m^{-1}, 1, m, m^2 \ldots$ *in direct order*, we multiply the roots of the polynomial by m. But also, if we use the terms of the series *in inverse order*, we divide the roots by m.

(44.) So, the first figure of the root now to be developed being in the tens' place, we commence by dividing the roots by 10, that we may use this first figure in the transformation as if it belonged to the unit's place.

(45.) The coefficients are

$$2 \qquad -25 \qquad -4 \qquad 50;$$

and using as multipliers 10^2, 10, 1, 10^{-1}, they are changed as desired, and the operation is as follows:—

200	−250	−4	5	(12·5
	−50	−54	−490	
	150	96	142000	
$2\cancel{0}\cancel{0}$	$35\cancel{0}$	174	0	
	39	26000		
	43	28400		
	470			

(46.) The first transformation (by 1) here gives

$$200 \qquad 350 \qquad 96 \qquad -49;$$

and we now *multiply* the roots by 10 by using 10^{-2}, 10^{-1}, 1, and 10, as multipliers, which gives

$$2 \qquad 35 \qquad 96 \qquad -490.$$

We then proceed as usual; and the absolute term disappearing, we know that 12·5 is a root (19, 22).

(47). It deserves to be noticed, that the effect of the first transformation is to increase the absolute term and introduce in it a change of sign: from 5 it has become −49. This is an indication that a root has been passed over, situated in the interval [0, 10], which is the root developed in (30).

(48.) Mr. Horner's paper, giving an account of his method of solving equations, was published in the *Philosophical Transactions* for 1819.* The attention of mathematicians was first specially called to the method by Professor J. R. Young, in his *Algebra*, published about 1823. The same gentleman has also, in several subsequent works—one of which is referred to in the Note appended to (29)—amply explained and illustrated it. The late Peter Nicholson published several tracts on the subject of it, putting forward at the same time a mistaken claim to the origination of it; and it was introduced by the late T. S. Davies into the second volume of his edition of *Hutton's Course*. A history of the Problem of Evolution from Vieta to Horner, which contains many interesting details, was contributed by Professor De Morgan to the *Companion to the Almanac* for 1839; and the subject is continued and amplified, and many illustrations given, in the article "Involution and Evolution," in the *Penny Cyclopædia* and the *Supplement* to the same work. These two have since been combined in the article under the same title in the *English Cyclopædia* ("Arts and Sciences").

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* It was republished in the Appendix to the *Ladies' Diary* for 1838. It must be admitted to be in an unattractive and needlessly transcendental form.

(49). Till very recently, when, as I am informed, it has been introduced by Mr. Todhunter in his *Theory of Equations*, Mr. Horner's method has been entirely ignored by all the University writers. It is not for me to suggest a reason for this; but it is certainly, in the circumstances, not a little singular to find retained in the text books the antiquated and laborious methods of transformation and solution, instead of Horner's elegant and efficient processes.

On the Construction of Tables of Mortality. *By* W. S. B. Woolhouse, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 30th April, 1866.]

A TABLE of mortality is designed to represent the number of lives which, according to the best deductions from past experience, may be expected to survive at the termination of each successive year of age, supposing these survivors to be derived exclusively from a certain number of persons originally taken either at birth or at a given age. As a practical index of the mathematical law of average mortality such a table may be taken as a trustworthy guide for the future, provided that the number of lives which enter into its formation be sufficiently large and the particulars respecting them be correctly registered; and, what is equally important, that the observations shall extend over a considerable number of years.

Whether theoretically or practically considered, a table of mortality presents, in the most simple, complete and convenient form, the elementary data requisite for all computations involving the contingencies of life, and it is, therefore, universally adopted as the primitive basis upon which the whole superstructure of life assurance is built. In fact, it is the general permanence in this progressive distribution of mortality, between narrow probable limits of divergence, that constitutes the main stability and efficiency of Life Assurance Institutions. It is consequently needless to insist on the importance of accumulating reliable facts as to the statistics of human life, whenever that can be done on a large and efficient, scale, with the view of obtaining more perfect tables, or of more strictly testing the general accuracy of those in ordinary use, upon which an enormous amount of monied transactions is necessarily made to depend.

In the year 1839 I had occasion, officially, to go into an investigation of the mortality in the Indian army. An abstract of this investigation was published at the time in the form of a pamphlet. As discussions of this kind are, as a matter of course, attended by a somewhat tedious amount of numerical operations, I was induced, in the first instance, to examine the mathematical relations with the object of systematically reducing them to such a form as would give the utmost brevity and simplicity to the process of calculation, and, what is of still greater moment, obviate any possible risk of error. The resulting method which I then followed was afterwards approved and expressly adopted by the Committee of Actuaries in the computation of the "Tables exhibiting the Law of Mortality, deduced from the Combined Experience of Seventeen Life Assurance Offices;" and the superintendence of those calculations was eventually entrusted to the author, who was a member of that committee. In the present paper it is principally intended, amongst other matters relating to the same subject, to give a short exposition of this particular method, since but a cursory outline of it appeared in the pamphlet referred to; and, with a view to the practical utility of this object, it may be premised that the inquiry will admit of the most satisfactory and comprehensive application, if we suppose the investigation to appertain to the registered experience of a Life Office, since in this case the returns may be presumed to supply complete information with respect to each individual life.

For an accurate discussion of this kind the essential materials of information with respect to each life are—

1. The Office age on entry, or the date of birth;
2. The date of admission;
3. The date of exit, if such event has taken place;
4. The cause of such exit, whether by discontinuance or by death.

We shall, however, for the occasion of presenting an example in numbers, first consider the subject under ordinary conditions, in which, for the purpose of facility and simplicity, the particulars are not given with the exactitude here prescribed. Afterwards it will be easy to show what modification will be needed, if in any investigation the information were accurately defined and greater precision were required in the method of computation.

In the "Experience of Offices" only the calendar years, instead of the dates of entry and exit, were deemed to be sufficient for the purpose, it being considered that large numbers, promiscuously spread over any given year, when taken in combination, may on the average, be practically assumed with tolerable correctness as uniformly dating from the middle of the year. For a like reason, only the current year of age, or the Office age, is required to be stated. Thus, by assuming the number of lives that entered in each current year of age to be, on the average, admitted also at the middle of such year a convenient coincidence will thereby subsist between calendar years and years of age; and the lives, one with another, may be considered as completing the Office age at the end of the calendar year of entry. The collection of the original data was therefore made in the following form:—

<table>
<tr><th colspan="10">CONTAINING THE EXPERIENCE OF THE OFFICE UP TO THE END OF THE YEAR 18 .</th></tr>
<tr><th rowspan="2">For use of Office.</th><th rowspan="2">Current Age at Entry.</th><th colspan="2">Year of</th><th rowspan="2">If by Death, D.</th><th rowspan="2"></th><th rowspan="2">Sex, if Female, F.</th><th rowspan="2">Distinction into Town, T. Country, C. Irish, I.</th><th rowspan="2">Cause of Death.</th><th rowspan="2">Special Risks and Remarks.</th></tr>
<tr><th>Entry.</th><th>Exit.</th></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

When the list, according to this form, is made out and completed, the column containing the year of exit will at once disclose the policies that have ceased to exist through *discontinuance* or *death* during the period. The number of years they were severally in force, or the duration of each of these policies, will be deduced by taking the difference between the year of entry and the year of exit; and the current age of exit will be found by adding the years of duration to the current age at entry.

The policies which do not exhibit any year of exit are those which remained in force, and the lives of which were in *existence* at the end of the period of observation. In like manner, the years of duration of these policies will be found by taking the difference between the year of entry and the year to which the investigation extends, as stated at the top of the list. Also by adding, as before, the duration to the current age at entry we get, in each case, the current age to which the life attained and existed at the end of the period.

So far all the requisite details of the facts are separately known, and are ready for the purely mechanical work of classification and condensation. The chief objects to be accomplished will be to resolve the whole of the policies into three divisions—viz., 1, policies DISCONTINUED without death; 2, policies EXISTING at the termination of the period; and 3, policies cancelled by DEATH; which three divisions, we think, it will be found most convenient to give separately; from these to bring together all policies having the same age of entry; and, finally, to arrange and enumerate them, either with respect to the years of duration or according to the ages attained. By this means the whole of the facts will become moulded into three tables, of which the following will serve as a specimen :—

TABLE (A).—*Enumeration of Discontinuances.*

Current Age of Entry	Year of Duration of Policy																			Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
4	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	1
5	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	1
6	..	1	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	4
7	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	2
8	..	2	1	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	5
9	..	2	..	2	3	..	..	4	..	1	..	1	..	..	..	..	..	..	„ ..	13
10	..	4	3	4	1	1	..	2	..	2	..	..	1	..	..	..	..	..	„ ..	18
11	..	4	2	2	..	3	..	1	1	1	..	..	..	..	..	..	..	..	„ ..	14
12	..	3	2	4	1	2	..	4	..	..	1	..	..	..	..	..	..	..	„ ..	17
13	..	11	5	1	1	..	..	2	1	1	..	..	..	..	..	..	..	..	„ ..	22
14	..	4	3	3	2	1	2	3	2	1	..	2	..	..	..	..	..	..	„ ..	23
15	..	8	5	2	3	3	..	5	..	..	1	..	..	..	..	..	..	..	„ ..	27
16	1	6	6	2	3	1	1	2	..	1	..	..	..	..	..	..	..	..	„ ..	23
17	..	10	6	4	4	2	1	8	..	..	1	..	..	..	..	..	..	..	„ ..	36
18	..	8	3	12	10	4	3	1	..	..	..	..	..	..	..	..	..	..	„ ..	41
19	..	15	17	21	5	1	..	8	..	2	..	..	..	..	..	..	..	..	„ ..	69
20	1	38	59	9	6	8	2	4	..	..	..	..	..	..	..	..	..	..	„ ..	127
&c.																				&c.

TABLE A.—*Enumeration of Discontinuances.*

Current Age of Entry	Current Age attained																			Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
4	—	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	1
5	—	—	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	1
6	—	—	—	..	1	1	1	1	..	..	..	..	..	..	..	..	..	..	„ ..	4
7	—	—	—	—	..	2	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	2
8	—	—	—	—	—	..	2	1	1	..	1	..	..	..	..	..	..	..	„ ..	5
9	—	—	—	—	—	—	..	2	..	2	3	..	..	4	..	1	..	1	„ ..	13
10	—	—	—	—	—	—	—	..	4	3	4	1	1	..	2	..	2	..	„ ..	18
11	—	—	—	—	—	—	—	—	..	4	2	2	..	3	..	1	1	1	„ ..	14
12	—	—	—	—	—	—	—	—	—	..	3	2	4	1	2	..	4	..	„ ..	17
13	—	—	—	—	—	—	—	—	—	—	..	11	5	1	1	..	..	2	„ ..	22
14	—	—	—	—	—	—	—	—	—	—	—	..	4	3	3	2	1	2	„ ..	23
15	—	—	—	—	—	—	—	—	—	—	—	—	..	8	5	2	3	3	„ ..	27
16	—	—	—	—	—	—	—	—	—	—	—	—	—	1	6	6	2	3	„ ..	23
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	10	6	4	„ ..	36
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	8	3	„ ..	41
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	15	„ ..	69
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	„ ..	127
&c.																				&c.
(*r*)	..	..	..	..	3	3	3	4	5	9	13	16	14	21	19	22	27	35	&c.	8,188

TABLE (B).—*Enumeration of Policies finally Existing.*

Current Age of Entry	Year of Duration of Policy																			Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	” ..	1
4	..	1	1		1	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	3
5	..	..	..	3	1	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	4
6	1	1	2	2	2	..	..	..	..	..	..	..	1	..	..	..	..	..	” ..	9
7	1	3	1	3	1	2	1	..	..	..	1	2	1	..	..	..	..	..	” ..	16
8	..	2	3	..	3	1	1	..	2	1	2	1	..	..	..	..	..	..	” ..	16
9	5	1	2	4	1	3	1	2	3	1	..	1	2	..	..	..	..	..	” ..	26
10	10	2	3	3	6	3	1	3	1		..	2	1	2	..	..	..	..	” ..	37
11	2	1	..	3	4	..	3	3	1	3		..	2	2	1	..	..	..	” ..	25
12	8	2	2	4	6	1	4	4	..	1	1	1	2	..	1	..	..	..	” ..	37
13	8	4	4	4	6	4	..	2	2	1	1	3	2	..	..	..	..	..	” ..	41
14	5	1	4	6	6	6	3	3	5	2	2	1	1	1	1	..	1	..	” ..	48
15	8	7	4	3	6	1	2	6	..	1	..	..	5	..	..	..	1	..	” ..	44
16	7	2	5	14	4	7	2	4	1	5	2	2	6	5	2	..	1	..	” ..	69
17	8	16	10	14	6	5	3	..	2	1	..	1	3	2	1	..	..	1	” ..	73
18	13	13	3	9	9	4	3	2	3	4	2	1	3	1	..	..	..	..	” ..	70
19	18	9	8	8	4	7	1	..	..	5	2	2	1	1	5	..	..	..	” ..	71
20	14	18	7	8	4	7	6	1	3	4	3	3	4	5	..	..	..	1	” ..	88
&c.																				&c.

TABLE B.—*Enumeration of Policies finally Existing.*

Current Age of Entry	Current Age attained																			Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	” ..	1
4	—	..	1	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	” ..	3
5	—	—	..	..	..	3	1	..	..	..	..	..	..	..	..	..	..	..	” ..	4
6	—	—	—	1	1	2	2	2	..	..	..	..	..	..	..	1	..	..	” ..	9
7	—	—	—	—	1	3	1	3	1	2	1	..	..	..	1	2	1	..	” ..	16
8	—	—	—	—	—	..	2	3	..	3	1	1	..	2	1	2	1	..	” ..	16
9	—	—	—	—	—	—	5	1	2	4	1	3	1	2	3	1	..	1	” ..	26
10	—	—	—	—	—	—	—	10	2	3	3	6	3	1	3	1	..	..	” ..	37
11	—	—	—	—	—	—	—	—	2	1	..	3	4	..	3	3	1	3	” ..	25
12	—	—	—	—	—	—	—	—	—	8	2	2	4	6	1	4	4	..	” ..	37
13	—	—	—	—	—	—	—	—	—	—	8	4	4	4	6	4	..	2	” ..	41
14	—	—	—	—	—	—	—	—	—	—	—	5	1	4	6	6	6	3	” ..	48
15	—	—	—	—	—	—	—	—	—	—	—	—	8	7	4	3	6	1	” ..	44
16	—	—	—	—	—	—	—	—	—	—	—	—	—	7	2	5	14	4	” ..	69
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	16	10	14	” ..	73
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13	13	3	” ..	70
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18	9	” ..	71
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14	” ..	88
&c.																				&c.
(R)	..	..	1	2	2	10	11	19	7	21	16	24	25	33	38	61	74	54	&c.	20,710

TABLE (C).—*Enumeration of Deaths.*

Current Age of Entry	Year of Duration of Policy																			Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
5	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
7	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	” ..	1
8	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
10	..	..	..	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	” ..	2
11	..	..	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	3
12	..	..	..	..	1	..	..	1	..	..	..	..	..	..	..	..	..	..	” ..	2
13	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
14	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	” ..	2
15	1	..	1	..	..	1	..	..	1	..	..	..	..	..	..	..	..	..	” ..	4
16	..	1	..	..	1	..	..	..	..	1	..	..	..	..	..	..	..	..	” ..	3
17	..	..	1	..	2	2	..	1	..	..	..	..	..	..	..	..	..	..	” ..	6
18	..	3	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	6
19	1	..	1	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	” ..	3
20	..	5	5	1	2	..	1	1	2	..	..	..	..	..	..	..	..	..	” ..	17
&c.																				&c.

TABLE C.—*Enumeration of Deaths.*

Current Age of Entry	Current Age attained																			Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
4	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
5	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
6	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
7	—	—	—	—	..	..	..	..	..	..	..	1	..	..	..	..	..	..	” ..	1
8	—	—	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
9	—	—	—	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	” ..	..
10	—	—	—	—	—	—	—	..	..	..	1	..	1	..	..	..	..	..	” ..	2
11	—	—	—	—	—	—	—	—	..	..	1	1	1	..	..	..	..	..	” ..	3
12	—	—	—	—	—	—	—	—	—	..	..	..	..	1	..	..	1	..	” ..	2
13	—	—	—	—	—	—	—	—	—	—	..	..	..	..	..	..	..	..	” ..	..
14	—	—	—	—	—	—	—	—	—	—	—	..	1	..	..	..	1	..	” ..	2
15	—	—	—	—	—	—	—	—	—	—	—	—	1	..	1	..	..	1	” ..	4
16	—	—	—	—	—	—	—	—	—	—	—	—	—	..	1	..	..	1	” ..	3
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	..	1	..	” ..	6
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	3	1	” ..	6
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	..	” ..	3
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	” ..	17
&c.																				&c.
(D)	..	..	..	..	..	..	..	..	..	..	2	2	4	1	2	..	7	3	&c.	2,482

The numbers here tabulated separately in the manner prescribed correctly represent, as far as they go, a combination of the four principal "Experience Tables," viz.:—

A(1), Male lives—Town, comprising . . .	16,097 lives
A(2), Male lives—Country „ . . .	11,926 „
A(4), Female lives—Town „ . . .	1,448 „
A(5), Female lives—Country „ . . .	1,909 „
Total	31,380 lives

The upper set of tables, (A), (B), (C), in which the numbers are distributed according to the years of duration of the policies, will have as many columns as the number of years over which the observations extend. When this period embraces only a limited number of years, the same being less than the number of ages, it will then be the most compact form in which to present the experience of an Office.

The lower set of tables, A, B, C, will have as many columns as may be sufficient to include all ages, and the contents will evidently consist of the same numbers as the former set; the only difference being, that the numbers which occupy the several horizontal lines are successively translated a place further to the right.

It will not be necessary in any case to prepare both of these sets of tables, as the data for any subsequent calculation can be readily extracted from either of them as they stand. The numbers we shall have occasion to combine will be those which appertain to the same acquired ages, and will, in the upper tables, ascend diagonally from the numbers immediately contiguous to each age of entry on the left; whereas in the more elongated tables, A, B, C, they will stand somewhat more conveniently in vertical columns. From sets of tables, in either of these forms, any number of which, representing different experiences, might be incorporated by simple addition, the various questions which relate to the value of selection of assured lives may be determined. If required, a table of mortality might be constructed for each separate age of entry, or for any distinct groups of ages; but it is needless to add, that such inquiries would be futile with small numbers.

The totals contained in the last column of each table are the sums of the numbers entered in each horizontal line when the table is completed. These numbers, collected from the three tables, will give the total number that entered in each current year of age as follows:—

Number that entered in each Age.

Current Age of Entry	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Total No. Disc.	..	1	1	4	2	5	13	18	14	17	22	23	27	23	36	41	69	127
„ „ Existing	1	3	4	9	16	16	26	37	25	37	41	48	44	69	73	70	71	88
„ „ Deaths.........		..		..	1	..	..	2	3	2	..	2	4	3	6	6	3	17
Total No. Issued (n)	1	4	5	13	19	21	39	57	42	56	63	73	75	95	115	117	143	232

For the computation of a table of mortality from the entire experience, which is, indeed, the main object in view, only the totals of the tables A, B, C will be needed; and if such a table of mortality should be the sole subject of inquiry, these totals might otherwise be most expeditiously found from the original returns* by filling in the ages of exit in the blank column and then simply counting the number of policies which answer to each of the several ages of entry; and afterwards counting, in like manner, the number of policies which answer to each age of exit, only remembering to distinguish these latter in the three divisions, so as to obtain separately the numbers (r), (R), and (D). For the purpose of following out the example already introduced, a complete statement of these totals is here tabulated.†

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* *See* the Form on page 77.

† An increased facility has recently been given to the processes of analysing the facts of a mortality experience by simply conveying the original particulars, with respect to each individual life, upon a separate card, specially designed to receive them. By this means the object of classifying each particular element, whatever it may be, is accomplished by the purely mechanical operation of shuffling the cards into the required order, without incurring the necessity, in any case, of making preliminary extracts of the several details. This is, undoubtedly, the most prominent advantage obtained, as there is always a much greater liability to error in making long and tedious transcriptions than in performing the operations of calculation. The plan, here briefly described, was first made publicly known in "An Account of the Processes employed in getting out the Mortality Experience of the Economic Life Assurance Society," and was the suggestion of Mr. O. G. Downes, of that Office. An appropriate and well-considered form of card, calculated to meet every requirement and convenience, is presented in the pamphlet referred to, and the method is appropriately designated the "card system." It is also pointed out that "it affords the means of making any deductions required, as it is only necessary to throw the cards into hotch-potch, and then rearrange them in the order needed to accomplish the required object—a process which would afford pleasant fireside amusement to any domesticated actuary and his family." The only disadvantage that could attend the method would appear to be the possibility of some of the cards, in their dignified isolation and independence, getting accidentally astray from the general mass, and carrying with them their separate contents, a casualty which could not happen when the materials are permanently entered on sheets, prepared for their reception. The mere idea of this supposed objection will, at least, induce a due amount of caution in those who, in the discussion of any branch of statistics, may be disposed to avail themselves of the manifold advantages of the card system.

TABLE D.—*Total Enumerations in each Year of Age.*

Current Year of Age	Entered (*n*)	Discontinued (*r*)	Existing at the Termination (R)	Died (D)	Current Year of Age	Entered (*n*)	Discontinued (*r*)	Existing at the Termination (R)	Died (D)
						23,307	5,734	11,608	1,005
3	1	..	..	..	46	667	204	694	70
4	4	..	..	..	47	703	206	695	54
5	5	..	1	..	48	640	178	591	66
6	13	..	2	..	49	657	182	569	79
7	19	3	2	..	50	593	152	536	57
8	21	3	10	..	51	489	166	513	57
9	39	3	11	.	52	522	168	495	56
10	57	4	19	..	53	448	134	486	59
11	42	5	7	..	54	368	134	442	64
12	56	9	21	..	55	373	144	421	61
13	63	13	16	2	56	359	93	387	76
14	73	16	24	2	57	329	88	358	64
15	75	14	25	4	58	302	89	324	48
16	95	21	33	1	59	267	79	277	65
17	115	19	38	2	60	245	72	288	56
18	117	22	61	..	61	183	57	312	55
19	143	27	74	7	62	178	43	276	49
20	232	35	54	3	63	159	57	212	37
21	286	83	78	11	64	131	38	196	45
22	389	219	92	13	65	119	22	169	61
23	453	102	115	6	66	85	35	127	39
24	509	119	131	11	67	79	25	148	35
25	722	184	183	10	68	47	32	99	33
26	710	158	181	14	69	28	15	89	28
27	878	188	266	16	70	26	9	84	25
28	910	212	286	28	71	20	7	98	27
29	1,000	219	332	24	72	14	4	40	15
30	1,083	210	339	22	73	14	4	48	14
31	1,059	243	379	31	74	11	2	32	15
32	1,073	248	428	41	75	5	7	23	11
33	1,123	255	489	27	76	3	2	16	19
34	1,175	269	555	57	77	3	1	16	6
35	1,145	271	591	51	78	3	2	12	10
36	1,142	266	639	67	79	2	..	16	11
37	1,089	285	616	49	80	..	1	6	3
38	1,027	266	732	67	81	..	1	1	2
39	967	271	667	70	82	..	..	..	..
40	1,055	264	709	43	83	..	..	..	..
41	904	272	711	71	84	1	..	3	1
42	901	266	679	68	85	..	1	2	3
43	887	235	683	54	86	..	..	..	1
44	849	234	693	71	87	..	..	..	..
45	801	201	636	62	88	..	..	1	..
	23,307	5,734	11,608	1,005	Total	31,380	8,188	20,710	2,482

The data thus prepared, on which the calculation of the required table of mortality is to be founded, embody a complete analysis and classification of a mass of observations which extends over a given number of years. To proceed with the calculation of the the rates of mortality the principal requirement will be to ascertain how many lives, during that period, have entered and passed through each year of age, so as to be enabled to compare the same with the number of deaths that have respectively occurred in the same years. It will also appear that these estimates will not be affected by any chronological considerations, since a question as to when any specified life or lives entered the year of age is quite immaterial and forms no part of the inquiry. We only require to know how many lives have passed through a proposed year of age at all the various times during the period of observation, but without any reference to those times.

In any proposed current year of age let

N denote the number of lives in existence at the commencement of the period;

$$\left.\begin{matrix} n \\ r \\ D \end{matrix}\right\} \text{the number of} \left\{\begin{matrix} \text{admissions} \\ \text{retirements} \\ \text{deaths} \end{matrix}\right\} \text{in the course of the period;}$$

R the number remaining in existence on the list at the termination of the period;

ϵ the total number that, in the course of the period, have passed the point of commencement of the proposed year of age, or the total number that have entered as survivors at the commencement; and

N_1, n_1, r_1, D_1, R_1, ϵ_1, the like numbers for the next higher year of age, &c.

Then it is evident that the number of lives which have existed at the end of the year of age may be directly deduced from the number at the commencement, by merely taking into account the several operations or events that have occurred in the same year. Thus, at the end of the proposed year of age, the survivors ϵ, at the beginning of that year, will, in the course of the year of age, become augmented by $N+n$ and diminished by $R+r+D$. The resulting number that existed at the end of the year is therefore $\epsilon+(N+n)-(R+r+D)$, and this must necessarily be the number ϵ_1 that enter by survivorship the next higher year of age. Whence, if

$$\left.\begin{aligned} \theta &= (N+n)-r \\ \omega &= (N+n)-(R+r) \\ &= \theta - R \end{aligned}\right\} \quad . \quad . \quad . \quad (1),$$

then will
$$\varepsilon_1 = \varepsilon + \omega - D,$$
$$\left.\begin{array}{r} \text{or } \Delta\varepsilon = \omega - D \\ \varepsilon_1 = \varepsilon + \Delta\varepsilon \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This simple and obvious deduction being applied consecutively to each retrogressive year of age will put us in possession of the subsisting relations which determine and guide the calculation.

Thus, by commencing at any age x, the several relations put down in a descending order are

$$\begin{aligned} \varepsilon_x &= \varepsilon_{x-1} + \omega_{x-1} - D_{x-1} \\ &= \varepsilon_{x-1} + (\omega - D)_{x-1} \\ \varepsilon_{x-1} &= \varepsilon_{x-2} + (\omega - D)_{x-2} \\ \varepsilon_{x-2} &= \varepsilon_{x-3} + (\omega - D)_{x-3} \\ \&c. &\qquad \&c. \end{aligned}$$

Hence, by successive substitution, or by adding the equations together and cancelling identities, which leads to the same result, we obtain

$$\begin{aligned} \varepsilon_x &= (\omega - D)_{x-1} + (\omega - D)_{x-2} + (\omega - D)_{x-3} + \&c. \\ &= \Sigma_{x-1}(\omega - D) = \Sigma_{x-1}(\Delta\varepsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3), \end{aligned}$$

in which it will be seen that the characteristic Σ, as indicated, includes the accumulation of values for all the years of age which precede the year under consideration. And thus we find, by continued summation, the total number of lives that from time to time, in the course of the period, have existed as survivors at the commencement of each successive year of age.

Now, according to the blank form specified on page 77, in which the original returns were entered, the several numbers representing the experience are inclusive of the whole duration of the initial and terminal years of observation; and it will thence follow that the number R, finally in existence, must be regarded as appertaining to the end of the corresponding year of age. But the numbers N, *n*, *r*, which compose θ, or, in other words, the several entrances and exits which, in the year of age, have operated in disturbing the number of lives otherwise than by death are distributed through the year of age at indiscriminate intervals and may, with large numbers, be practically considered, on the average, as having reference to the middle of the year, or as having undergone half the chance of death in one year.* The RISK may therefore be regarded as that of $\varepsilon + \frac{1}{2}\theta$

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* The average interval by which the date of entry precedes the birthday is here assumed to be half a year, which may be expected to be somewhat in excess, since in some cases it is natural to surmise a disposition to expedite an assurance when a coming birthday, in near perspective, is suggestive of a higher rate of premium. According to actual experience the average interval is found to be about three-eighths of a year. The

persons entering upon the year, and the uninterrupted survivors of the same continuing exposed to its termination. Therefore, as D is the number of deaths that have occurred in the same year of age, the corresponding number of survivors may be considered to be $\epsilon+\frac{1}{2}\theta-D$. Hence the proportionate rate of mortality in the year is

$$m=\frac{D}{\epsilon+\frac{1}{2}\theta};$$

and the proportion that survive, or the probability of surviving the year is

$$p=\frac{\epsilon+\frac{1}{2}\theta-D}{\epsilon+\frac{1}{2}\theta}.$$

To abbreviate, let

$$\left.\begin{aligned}\beta&=\epsilon+\tfrac{1}{2}\theta\\ \gamma&=\beta-D\\ \text{then } m&=\frac{D}{\beta}\\ p&=\frac{\gamma}{\beta}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4).$$

Also if, according to the usual notation, l_x, l_{x+1}, &c., denote the numbers living in the required table of mortality at the ages x, $x+1$, &c., the logarithms of these numbers for all ages in the table are hence deduced from the logarithms of p, by successive addition, beginning with a suitable number as an arbitrary radix for the first age of the table. Thus,

$$\begin{aligned}\log l_{x+1}&=\log p_x \quad +\log l_x\\ \log l_{x+2}&=\log p_{x+1}+\log l_{x+1}\\ \&c. & \qquad\qquad \&c.\end{aligned}$$

The following example, worked out at length, will perhaps best exemplify the practical application of the preceding formulæ. The data are taken from Table D, observing that N=0 for all ages, since the returns record every life from the year of entry.

As the numbers for the younger ages are small the mortality table is made to begin with age 10, and the radix is $l_{10}=100000$.

For the latter ages some of the numbers come out negative, and it then becomes requisite to attend to the algebraic signs.

The values of ϵ are completed horizontally by successive addition, and the final vanishing of these numbers after the last age of the table will, up to this point, supply a check on the accuracy of the calculation.

difference is however not material. As the lives are by this means accounted nearly one-eighth of a year younger than they really are, the ultimate tendency will be to slightly increase the rate of mortality. A more accurate method of computation is given further on.

The values of log l are also completed horizontally in a similar manner.

In these calculations it will be observed that the annual rate of mortality, m, is given under its usual acceptation, viz.:—

$$m_x = \frac{l_x - l_{x+1}}{l_x} = \frac{\text{decrement}}{\text{number living}}.$$

Completed Age	2	3	4	5	6	7	8	9
$N+n$	1	4	5	13	19	21	39	57
r	..	..	..	..	3	3	3	4
$(N+n)-r=\theta$	1	4	5	13	16	18	36	53
R	..	..	1	2	2	10	11	19
$\theta-R=\omega$	1	4	4	11	14	8	25	34
D	..	..	..	..	..	..	..	..
$\omega-D=\Delta\varepsilon$	1	4	4	11	14	8	25	34
ε	0	1	5	9	20	34	42	67
$\frac{1}{2}\theta$	0·5	2·0	2·5	6·5	8·0	9·0	18·0	26·5
$\varepsilon+\frac{1}{2}\theta=\beta$	0·5	3·0	7·5	15·5	28·0	43·0	60·0	93·5
$\beta-D=\gamma$	0·5	3·0	7·5	15·5	28·0	43·0	60·0	93·5

Age	10	11	12	13	14
$N+n$	42	56	63	73	75
r	5	9	13	16	14
θ	37	47	50	57	61
R	7	21	16	24	25
ω	30	26	34	33	36
D	..	..	2	2	4
$\Delta\varepsilon$	30	26	32	31	32
ε	101	131	157	189	220
$\frac{1}{2}\theta$	18·5	23·5	25·0	28·5	30·5
β	119·5	154·5	182·0	217·5	250·5
γ	119·5	154·5	180·0	215·5	246·5
$\log\beta$	2·07737	2·18893	2·26007	2·33746	2·39881
$\log\gamma$	2·07737	2·18893	2·25527	2·33345	2·39182
$\log p$	0·00000	0·00000	9·99520	9·99599	9·99301
$\log l$	5·00000	5·00000	5·00000	4·99520	4·99119
l	100000	100000	100000	98901	97992
p	1·00000	1·00000	·98901	·99081	·98403
m	·00000	·00000	·01099	·00919	·01597

Age	15	16	17	18	19
$N+n$	95	115	117	143	232
r	21	19	22	27	35
θ	74	96	95	116	197
R	33	38	61	74	54
ω	41	58	34	42	143
D	1	2	0	7	3
$\Delta\varepsilon$	40	56	34	35	140
ε	252	292	348	382	417
$\frac{1}{2}\theta$	37·0	48·0	47·5	58·0	98·5
β	289·0	340·0	395·5	440·0	515·5
γ	288·0	338·0	395·5	433·0	512·5
$\log\beta$	2·46090	2·53148	2·59715	2·64345	2·71223
$\log\gamma$	2·45939	2·52892	2·59715	2·63649	2·70969
$\log p$	9·99849	9·99744	0·00000	9·99304	9·99746
$\log l$	4·98420	4·98269	4·98013	4·98013	4·97317
l	96427	96093	95528	95528	94009
p	·99653	·99412	1·00000	·98410	·99417
m	·00347	·00588	·00000	·01590	·00583

Age	20	21	22	23	24
$N+n$	286	389	453	509	722
r	83	219	102	119	184
θ	203	170	351	390	538
R	78	92	115	131	183
ω	125	78	236	259	355
D	11	13	6	11	10
$\Delta\varepsilon$	114	65	230	248	345
ε	557	671	736	966	1214
$\frac{1}{2}\theta$	101·5	85·0	175·5	195·0	269·0
β	658·5	756·0	911·5	1161·0	1483·0
γ	647·5	743·0	905·5	1150·0	1473·0
$\log\beta$	2·81856	2·87852	2·95976	3·06483	3·17114
$\log\gamma$	2·81124	2·87099	2·95689	3·06070	3·16820
$\log p$	9·99268	9·99247	9·99713	9·99587	9·99706
$\log l$	4·97063	4·96331	4·95578	4·95291	4·94878
l	93461	91899	90319	89724	88875
p	·98329	·98281	·99341	·99054	·99325
m	·01671	·01719	·00659	·00946	·00675

Age	25	26	27	28	29
$N+n$	710	878	910	1000	1083
r	158	188	212	219	210
θ	552	690	698	781	873
R	181	266	286	332	339
ω	371	424	412	449	534
D	14	16	28	24	22
$\Delta\epsilon$	357	408	384	425	512
ϵ	1559	1916	2324	2708	3133
$\frac{1}{2}\theta$	276·0	345·0	349·0	390·5	436·5
β	1835·0	2261·0	2673·0	3098·5	3569·5
γ	1821·0	2245·0	2645·0	3074·5	3547·5
$\log\beta$	3·26364	3·35430	3·42700	3·49115	3·55261
$\log\gamma$	3·26031	3·35122	3·42243	3·48777	3·54992
$\log p$	9·99667	9·99692	9·99543	9·99662	9·99731
$\log l$	4·94584	4·94251	4·93943	4·93486	4·93148
l	88275	87601	86982	86072	85404
p	·99236	·99293	·98953	·99225	·99383
m	·00764	·00707	·01047	·00775	·00617

Age	30	31	32	33	34
$N+n$	1059	1073	1123	1175	1145
r	243	248	255	269	271
θ	816	825	868	906	874
R	379	428	489	555	591
ω	437	397	379	351	283
D	31	41	27	57	51
$\Delta\epsilon$	406	356	352	294	232
ϵ	3645	4051	4407	4759	5053
$\frac{1}{2}\theta$	408·0	412·5	434·0	453·0	437·0
β	4053·0	4463·5	4841·0	5212·0	5490·0
γ	4022·0	4422·5	4814·0	5155·0	5439·0
$\log\beta$	3·60778	3·64968	3·68494	3·71700	3·73957
$\log\gamma$	3·60444	3·64567	3·68251	3·71223	3·73552
$\log p$	9·99666	9·99599	9·99757	9·99523	9·99595
$\log l$	4·92879	4·92545	4·92144	4·91901	4·91424
l	84877	84227	83453	82987	82080
p	·99234	·99081	·99442	·98908	·99072
m	·00766	·00919	·00558	·01092	·00928

Age	35	36	37	38	39
$N+n$	1142	1089	1027	967	1055
r	266	285	266	271	264
θ	876	804	761	696	791
R	639	616	732	667	709
ω	237	188	29	29	82
D	67	49	67	70	43
$\Delta\varepsilon$	+170	+139	−38	−41	+39
ε	5285	5455	5594	5556	5515
$\frac{1}{2}\theta$	438·0	402·0	380·5	348·0	395·5
β	5723·0	5857·0	5974·5	5904·0	5910·5
γ	5656·0	5808·0	5907·5	5834·0	5867·5
$\log\beta$	3·75762	3·76768	3·77630	3·77115	3·77162
$\log\gamma$	3·75251	3·76403	3·77140	3·76597	3·76845
$\log p$	9·99489	9·99635	9·99510	9·99482	9·99683
$\log l$	4·91019	4·90508	4·90143	4·89653	4·89135
l	81319	80367	79695	78801	77866
p	·98830	·99163	·98878	·98814	·99273
m	·01170	·00837	·01122	·01186	·00727

Age	40	41	42	43	44
$N+n$	904	901	887	849	801
r	272	266	235	234	201
θ	632	635	652	615	600
R	711	679	683	693	636
ω	−79	−44	−31	−78	−36
D	71	68	54	71	62
$\Delta\varepsilon$	−150	−112	−85	−149	−98
ε	5554	5404	5292	5207	5058
$\frac{1}{2}\theta$	316·0	317·5	326·0	307·5	300·0
β	5870·0	5721·5	5618·0	5514·5	5358·0
γ	5799·0	5653·5	5564·0	5443·5	5296·0
$\log\beta$	3·76864	3·75751	3·74958	3·74151	3·72900
$\log\gamma$	3·76335	3·75232	3·74539	3·73588	3·72395
$\log p$	9·99471	9·99481	9·99581	9·99437	9·99495
$\log l$	4·88818	4·88289	4·87770	4·87351	4·86788
l	77300	76364	75457	74733	73770
p	·98789	·98812	·99040	·98712	·98844
m	·01211	·01188	·00960	·01288	·01156

Age	45	46	47	48	49
$N+n$	667	703	640	657	593
r	204	206	178	182	152
θ	463	497	462	475	441
R	694	695	591	569	536
ω	−231	−198	−129	−94	−95
D	70	54	66	79	57
$\Delta\epsilon$	−301	−252	−195	−173	−152
ϵ	4960	4659	4407	4212	4039
$\frac{1}{2}\theta$	231·5	248·5	231·0	237·5	220·5
β	5191·5	4907·5	4638·0	4449·5	4259·5
γ	5121·5	4853·5	4572·0	4370·5	4202·5
$\log\beta$	3·71529	3·69086	3·66633	3·64831	3·62936
$\log\gamma$	3·70940	3·68605	3·66011	3·64053	3·62351
$\log p$	9·99411	9·99519	9·99378	9·99222	9·99415
$\log l$	4·86283	4·85694	4·85213	4·84591	4·83813
l	72917	71935	71143	70131	68886
p	·98653	·98899	·98578	·98225	·98662
m	·01347	·01101	·01422	·01775	·01338

Age	50	51	52	53	54
$N+n$	489	522	448	368	373
r	166	168	134	134	144
θ	323	354	314	234	229
R	513	495	486	442	421
ω	−190	−141	−172	−208	−192
D	57	56	59	64	61
$\Delta\epsilon$	−247	−197	−231	−272	−253
ϵ	3887	3640	3443	3212	2940
$\frac{1}{2}\theta$	161·5	177·0	157·0	117·0	114·5
β	4048·5	3817·0	3600·0	3329·0	3054·5
γ	3991·5	3761·0	3541·0	3265·0	2993·5
$\log\beta$	3·60729	3·58172	3·55630	3·52231	3·48494
$\log\gamma$	3·60114	3·57530	3·54913	3·51388	3·47618
$\log p$	9·99385	9·99358	9·99283	9·99157	9·99124
$\log l$	4·83228	4·82613	4·81971	4·81254	4·80411
l	67964	67009	66025	64944	63696
p	·98594	·98533	·98363	·98078	·98003
m	·01406	·01467	·01637	·01922	·01997

Age	55	56	57	58	59
$N+n$	359	329	302	267	245
r	93	88	89	79	72
θ	266	241	213	188	173
R	387	358	324	277	288
ω	−121	−117	−111	−89	−115
D	76	64	48	65	56
$\Delta\varepsilon$	−197	−181	−159	−154	−171
ε	2687	2490	2309	2150	1996
$\frac{1}{2}\theta$	133·0	120·5	106·5	94·0	86·5
β	2820·0	2610·5	2415·5	2244·0	2082·5
γ	2744·0	2546·5	2367·5	2179·0	2026·5
$\log\beta$	3·45025	3·41672	3·38301	3·35102	3·31859
$\log\gamma$	3·43838	3·40594	3·37429	3·33826	3·30675
$\log p$	9·98813	9·98922	9·99128	9·98724	9·98816
$\log l$	4·79535	4·78348	4·77270	4·76398	4·75122
l	62424	60741	59252	58074	56392
p	·97304	·97548	·98012	·97105	·97311
m	·02696	·02452	·01988	·02895	·02689

Age	60	61	62	63	64
$N+n$	183	178	159	131	119
r	57	43	57	38	22
θ	126	135	102	93	97
R	312	276	212	196	169
ω	−186	−141	−110	−103	−72
D	55	49	37	45	61
$\Delta\varepsilon$	−241	−190	−147	−148	−133
ε	1825	1584	1394	1247	1099
$\frac{1}{2}\theta$	63·0	67·5	51·0	46·5	48·5
β	1888·0	1651·5	1445·0	1293·5	1147·5
γ	1833·0	1602·5	1408·0	1248·5	1086·5
$\log\beta$	3·27600	3·21788	3·15987	3·11177	3·05975
$\log\gamma$	3·26316	3·20480	3·14860	3·09639	3·03603
$\log p$	9·98716	9·98692	9·98873	9·98462	9·97628
$\log l$	4·73938	4·72654	4·71346	4·70219	4·68681
l	54876	53277	51696	50372	48619
p	·97087	·97033	·97438	·96521	·94685
m	·02913	·02967	·02562	·03479	·05315

Age	65	66	67	68	69
$N+n$	85	79	47	28	26
r	35	25	32	15	9
θ	50	54	15	13	17
R	127	148	99	89	84
ω	−77	−94	−84	−76	−67
D	39	35	33	28	25
$\Delta\varepsilon$	−116	−129	−117	−104	−92
ε	966	850	721	604	500
$\frac{1}{2}\theta$	25·0	27·0	7·5	6·5	8·5
β	991·0	877·0	728·5	610·5	508·5
γ	952·0	842·0	695·5	582·5	483·5
$\log\beta$	2·99607	2·94300	2·86243	2·78569	2·70629
$\log\gamma$	2·97864	2·92531	2·84230	2·76530	2·68440
$\log p$	9·98257	9·98231	9·97987	9·97961	9·97811
$\log l$	4·66309	4·64566	4·62797	4·60784	4·58745
l	46035	44224	42459	40536	38677
p	·96066	·96009	·95471	·95414	·95085
m	·03934	·03991	·04529	·04586	·04915

Age	70	71	72	73	74
$N+n$	20	14	14	11	5
r	7	4	4	2	7
θ	13	10	10	9	−2
R	98	40	48	32	23
ω	−85	−30	−38	−23	−25
D	27	15	14	15	11
$\Delta\varepsilon$	−112	−45	−52	−38	−36
ε	408	296	251	199	161
$\frac{1}{2}\theta$	6·5	5·0	5·0	4·5	−1·0
β	414·5	301·0	256·0	203·5	160·0
γ	387·5	286·0	242·0	188·5	149·0
$\log\beta$	2·61752	2·47857	2·40824	2·30856	2·20412
$\log\gamma$	2·58827	2·45637	2·38382	2·27531	2·17319
$\log p$	9·97075	9·97780	9·97558	9·96675	9·96907
$\log l$	4·56556	4·53631	4·51411	4·48969	4·45644
l	36776	34380	32667	30881	28605
p	·93487	·95017	·94532	·92630	·93126
m	·06513	·04983	·05468	·07370	·06874

Age	75	76	77	78	79
$N+n$	3	3	3	2	..
r	2	1	2	..	1
θ	1	2	1	2	−1
R	16	16	12	16	6
ω	−15	−14	−11	−14	−7
D	19	6	10	11	3
$\Delta\epsilon$	−34	−20	−21	−25	−10
ϵ	125	91	71	50	25
$\tfrac{1}{2}\theta$	0·5	1·0	0·5	1·0	−0·5
β	125·5	92·0	71·5	51·0	24·5
γ	106·5	86·0	61·5	40·0	21·5
$\log\beta$	2·09864	1·96379	1·85431	1·70757	1·38917
$\log\gamma$	2·02735	1·93450	1·78888	1·60206	1·33244
$\log p$	9·92871	9·97071	9·93457	9·89449	9·94327
$\log l$	4·42551	4·35422	4·32493	4·25950	4·15399
l	26639	22606	21131	18176	14256
p	·84861	·93478	·86014	·78431	·87755
m	·15139	·06522	·13986	·21569	·12245

Age	80	81	82	83	84
$N+n$	..	..	..	1	..
r	1	..	..	..	1
θ	−1	..	..	+1	−1
R	1	..	..	3	2
ω	−2	..	..	−2	−3
D	2	..	..	1	3
$\Delta\epsilon$	−4	..	..	−3	−6
ϵ	15	11	11	11	8
$\tfrac{1}{2}\theta$	−0·5	..	..	+0·5	−0·5
β	14·5	11·0	11·0	11·5	7·5
γ	12·5	11·0	11·0	10·5	4·5
$\log\beta$	1·16137	1·04139	1·04139	1·06070	0·87506
$\log\gamma$	1·09691	1·04139	1·04139	1·02119	0·65321
$\log p$	9·93554	0·00000	0·00000	9·96049	9·77815
$\log l$	4·09726	4·03280	4·03280	4·03280	3·99329
l	12510	10784	10784	10784	9847
p	·86206	1·00000	1·00000	·91304	·60000
m	·13794	·00000	·00000	·08696	·40000

Age	85	86	87	88
$N+n$	..	..	..	..
r	...	..	...	..
θ	..	..	..	..
R	..	..	1	..
ω	..	..	-1	..
D	1	..	..	..
$\Delta\varepsilon$	-1	..	-1	..
ε	2	1	1	..
$\frac{1}{2}\theta$	..	..	..	..
β	2·0	1·0	1·0	..
γ	1·0	1·0	1·0	..
$\log\beta$	0·30103	0·00000	0·00000	..
$\log\gamma$	0·00000	0·00000	0·00000	..
$\log p$	9·69897	0·00000	0·00000	..
$\log l$	3·77144	3·47041	3·47041	..
l	5908	2954	2954	..
p	·50000	1·00000	1·00000	..
m	·50000	·00000	·00000	..

The computed values of the number living and of the rate of mortality per cent., for every year of age, are laid down in two drawings, so as to exhibit at one view the general character and consistency of the results. In the latter diagram is also shown the corresponding curve, according to the adjusted Experience Table. It will be found on examination that the irregularities that stand out so conspicuously, at the earliest and latest ages, may be traced to the smallness of the numbers (β) that enter into the calulation. The results would have been much more valuable if the original experience had extended over a longer period.

If the number living be taken as the element for final adjustment according to the rules given in a former paper (vol. xii., pages 140–1), the same aggregate tabular mortality or decrement must necessarily be retained between all points of actual coincidence, in whatever way the intermediate numbers may be modified; and this is certainly a legitimate and desirable condition. The minutiæ of accurate distribution, however important, ought to be a secondary consideration to that of a true embodiment of the total mortality. An expert method of effecting a tabular distribution of the rate of mortality, by a special application of Gompertz's formula, is given by Mr. Jellicoe in his valuable paper inserted in vol. ii., page 199, to which we shall have occasion to refer hereafter.

In the investigation of the experience of the Amicable Society, by the late Mr. Galloway, the ages on admission are assumed, as in the foregoing, to be half a year less than the Office ages; but in other respects the data are collected according to a succession of years, reckoned, in each case, from the exact date of entry as an epoch. The method of working these up will be a little different. Thus if we here employ notation, analogous to that which has preceded, the original numbers to be used in the computation are

$$\begin{array}{l} n' = \text{number of admissions at age } x-\tfrac{1}{2}; \\ \left.\begin{array}{l} R' = \text{number finally existing} \\ r' = \text{number discontinued} \\ D' = \text{number of deaths} \end{array}\right\} \begin{array}{c} \text{between the ages} \\ x-\tfrac{1}{2} \text{ and } x+\tfrac{1}{2}. \end{array} \end{array}$$

By taking the arithmetical mean values between each of these last numbers and the corresponding number for the preceding age, we shall then have the whole of the data, estimated in the way we have prescribed, viz.:—

$$\left.\begin{array}{ll} n = n' & = \text{number of admissions} \\ R = \tfrac{1}{2}(R'+R'_{-1}) & = \text{number finally existing} \\ r = \tfrac{1}{2}(r'+r'_{-1}) & = \text{number discontinued} \\ D = \tfrac{1}{2}(D'+D'_{-1}) & = \text{number of deaths} \end{array}\right\} \begin{array}{c} \text{in the } x\text{th year} \\ \text{of age.} \end{array}$$

Or, doubling to avoid fractions,

$$n=2n', \quad R=R'+R'_{-1}, \quad r=r'+r'_{-1}, \quad D=D'+D'_{-1}.$$

Hence, by taking these values, we can readily determine ε as before; but the number at risk will be $\varepsilon+\frac{1}{2}\omega$, instead of $\varepsilon+\frac{1}{2}\theta$, since the R lives finally existing are here assumed to be equally distributed over the year of age.

The implied hypothesis that lives admitted at ages of the form $x-\frac{1}{2}$ have their discontinuances equally distributed in years of the form $x-\frac{1}{2}$ to $x+\frac{1}{2}$ should be considered in the application of this method.

We shall now resume the original subject, with a more strict attention to minute details, and suppose the returns to supply the exact dates of birth, entry and exit. From these the precise ages at entry and exit can be stated to a fraction of a year; but the precise ages at death will not be required. The notation will be unchanged and the computation of the numbers ε, that enter the several years of age, will in every respect be the same. The modification to be made in the calculation will alone consist in the allowances to be made for the several entrances and exits which occur in the year of age, and which, being individually known, we are no longer constrained to take at an assumed average value. These entrances and exits when combined determine the number ω, viz.,

$$\omega=(N+n)-(R+r);$$

and it is now required to find the exact portions of the year of age that are occupied by the constituent numbers, N, n, R, r.

First take the N lives that exist at the commencement, in the current year of age x, and let the exact ages of these lives be severally $x-\delta N_1$, $x-\delta N_2$, $x-\delta N_3$, &c., and let the sum of the various fractions of years, δN_1, δN_2, δN_3, &c., be denoted by $\Sigma\delta N$.

Similarly, let the exact ages of the n lives admitted in the year of age be severally $x-\delta n_1$, $x-\delta n_2$, $x-\delta n_3$, &c., and let the sum of the fractions δn_1, δn_2, δn_3, &c., be $\Sigma\delta n$.

Again, let the exact ages of the R lives finally existing, in the year of age, be severally $x-\delta R_1$, $x-\delta R_2$, $x-\delta R_3$, &c., and let the sum of the fractions δR_1, δR_2, δR_3, &c., be $\Sigma\delta R$.

And, lastly, let the exact ages of the r discontinuances be $x-\delta r_1$, $x-\delta r_2$, $x-\delta r_3$, &c., and let the sum of the fractions δr_1, δr_2, δr_3, &c., be $\Sigma\delta r$.

Then the more accurate number of lives at risk will evidently be

$$\beta=\epsilon+(\Sigma\delta N+\Sigma\delta n)-(\Sigma\delta R+\Sigma\delta r).$$

The values of each set of fractions (δN, δn, δR, δr), may be put down either in decimals of a year or in days. In the latter case the sums would have to be divided by 365. The other steps of the calculation will be precisely the same as before.

In Table D the respective sums $\Sigma\delta N$, $\Sigma\delta n$, $\Sigma\delta R$, $\Sigma\delta r$, might be conveniently exhibited in columns contiguous to those which contain the corresponding integral numbers N, n, R, r.

It may be added that the principles of the calculation, as here given, are now as exact as might reasonably be required. The only hypothesis of an approximate nature is that the various fractional intervals at the commencement and at the termination of each year of age are assumed to be subject to the same rate of mortality as that which obtains for the entire year. If a recalculation were made the method of accurately apportioning these estimates, in accordance with the results previously found, would be too obvious to need any explanation here; but it can hardly ever be thought necessary to expend the resources of calculation upon such minute refinements.

Before concluding this paper it may not be out of place to make a few remarks on some of the existing methods of eradicating the irregularities of tabular results. Perhaps, the most prevalent assumption in these methods has been that of taking the arithmetical mean of a group of consecutive values and adopting the result as the adjusted value which belongs to the middle of the group. The practical operation of taking averages has an undoubted tendency to neutralize errors of a purely incidental kind; but unless the process be sound also in theory, it must at the same time be attended by the introduction of errors in principle. It is here proposed to inquire into the amount of these latter errors, which admit of being accurately calculated from a series of differences.

Referring to my paper on "Interpolation," vol. xi., page 83, "Central Formulæ," let the general term of a series of values be denoted by V_x and the differences as there stated, and we shall have, by the formula (γ),

$$\begin{aligned} V_n=(V)+(n-\tfrac{1}{2})a_1+\frac{(n-1)n}{2}\left((b)+\frac{n-\frac{1}{2}}{3}c_1\right) \\ +\frac{(n-2)(n-1)n(n+1)}{2.3.4}\left((d)+\frac{n-\frac{1}{2}}{5}e_1\right) \\ +\frac{(n-3)(n-2)(n-1)n(n+1)(n+2)}{2.3.4.5.6}\left((f)+\frac{n-\frac{1}{2}}{7}g_1\right)+\&c. \end{aligned}$$

$$\begin{aligned} \therefore\ V_{n+1}=(V)+(n+\tfrac{1}{2})a_1+\frac{n(n+1)}{2}\left((b)+\frac{n+\frac{1}{2}}{3}c_1\right) \\ +\frac{(n-1)n(n+1)(n+2)}{2.3.4}\left((d)+\frac{n+\frac{1}{2}}{5}e_1\right) \\ +\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6}\left((f)+\frac{n+\frac{1}{2}}{7}g_1\right)+\&c. \end{aligned}$$

$$\begin{aligned} V_{-n}=(V)-(n+\tfrac{1}{2})a_1+\frac{n(n+1)}{2}\left((b)-\frac{n+\frac{1}{2}}{3}c_1\right) \\ +\frac{(n-1)n(n+1)(n+2)}{2.3.4}\left((d)-\frac{n+\frac{1}{2}}{5}e_1\right) \\ +\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6}\left((f)-\frac{n+\frac{1}{2}}{7}g_1\right)+\&c. \end{aligned}$$

$$\begin{aligned} \therefore\ V_{n+1}-V_{-n}=(2n+1)a_1+\frac{n(n+1).(2n+1)}{2.3}c_1 \\ +\frac{(n-1)n(n+1)(n+2).(2n+1)}{2.3.4.5}e_1 \\ +\frac{(n-2)(n-1)n(n+1)(n+2)(n+3).(2n+1)}{2.3.4.5.6.7}g_1+\&c. \end{aligned}$$

Now, this must evidently be equal to the sum of the $2n+1$ intervening values of the first differences a, which lie between V_{-n} and V_{n+1}. Therefore, the arithmetical mean of these $2n+1$ values

$$=a_1+\frac{n(n+1)}{2.3}c_1+\frac{(n-1)n(n+1)(n+2)}{2.3.4.5}e_1$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6.7}g_1+\&c.$$

Hence, by analogy in the scheme of differences, we infer the general theorem that the arithmetical mean of $2n+1$ consecutive values of V

$$=V+\frac{n(n+1)}{2.3}b+\frac{(n-1)n(n+1)(n+2)}{2.3.4.5}d$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6.7}f+\&c.\ .\ .\ (h),$$

where V, b, d, f, &c., are central values in the difference scheme. As n is integral the expression becomes completed in $n+1$ terms.

Taking the case $n=2$, we find for the mean of five values the simple formula $V+b+\frac{1}{5}d$.

Therefore the error engendered by assuming the arithmetical mean of five values to replace the middle value is equal to the middle second difference together with one-fifth of the fourth difference; and as fourth differences are usually small the second difference alone will in general be a sufficient indication of the amount of error.

The following example, with five logarithms, is given as a practical illustration:—

No.	Logarithms.	Differences.				
9	0·95424					
			+4576			
10	1·00000			−437		
			4139		+77	
11	1·04139	V		−360b		−20d
			3779		+57	
12	1·07918			−303		
			+3476			
13	1·11394					

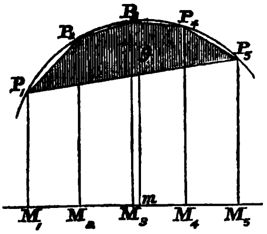
	5)5·18875		
Average of five values	1·03775	Second difference	−360
True middle value	1·04139	$\frac{1}{5}$ fourth difference	− 4
	−364	Error . . .	−364

That the relation here stated is rigorously exact, whatever be the quantities, may be otherwise shown, as follows:—

The five values being denoted by V_{-2}, V_{-1}, V, V_{+1}, V_{+2}, the central second and fourth differences are

$$\begin{aligned} b &= V_{-1}-2V+V_{+1} \\ d &= V_{-2}-4V_{-1}+6V-4V_{+1}+V_{+2} \\ \therefore b+\tfrac{1}{5}d &= \tfrac{1}{5}(V_{-2}+V_{-1}+V+V_{+1}+V_{+2})-V \\ &= \text{mean value}-\text{true middle value.} \end{aligned}$$

The existance of the error here pointed out becomes very apparent when considered geometrically. In the annexed diagram let the five ordinates, P_1M_1, P_2M_2, &c., represent the five values. Join the five points, $P_1P_2P_3P_4P_5$, throughout the circuit by straight lines, so as to form an irregular pentagon, which, in the diagram, is distinguished by the shading; and let g be the mean centre of the five points, or centre of gravity of the pentagon. Then, by geometry, it is well known that the arithmetical mean of the five ordinates is equal to the ordinate gm, demitted from the mean centre g of the pentagon, or of the five points P, and of course differs from the middle value P_3M_3. When the curve on which the points P are situated is concave downwards, as shown in the diagram, the error is obviously negative, and *vice versâ*.



In cases where the average is that of a group comprising a greater number of values than five, the magnitude of the error is of course greater, the formula (h) showing that it varies nearly as the square of the number of values that compose the group. When, therefore, as many as ten quantities are included, as in the formation of the English Life Table, and is necessarily so in everything derived from the population and registration returns for decennial periods, the discrepency involved by the stated assumption will amount to as much as four times the value of the central second difference.

It is not meant here to make any general exception to the simplifications obtainable by means of averages. A decided practical advantage, and a desirable reduction of casual imperfections, may often be gained by condensing an unadjusted series of results in groups; and such a process is, of course, perfectly legitimate, provided the several combinations are only represented to be what they really are, and treated accordingly. An instructive example of this kind may be found in the account of an investigation of the mortality experienced in the Eagle Insurance Company, given in vol. ii., page 199, in which Mr. Jellicoe, by grouping together the values of the mortality per cent. and discussing the combined results with great skill and judgment, succeeds in distributing them into convenient tabular values, without any displacement of the aggregate mortality contained in each of the respective groups.

The methods of adjusting the probabilities of life by means of Gompertz's formula,

$$\log l_x = \log k + \log g . q^x,$$

are usually carried out by breaking the table up into portions, commencing and terminating at certain stated ages, and calculating each of these portions from a distinct set of constants. The results of all such calculations necessarily have the defect that the different component curves do not unite continuously at the points of juncture.*

In a former paper, "on Gompertz's Law of Mortality," I propounded certain generalizations under which a continuous curve might be generated, in the following manner:—"As the formula contains three arbitrary constants it will follow that if the calculated mortality be laid down in a curve, it may be made to pass through any three assigned points, and that the assumption of three such points will be sufficient to determine the values of the constants. But in a curve of mortality that does not conform with Gompertz's law, the computed values of the constants will vary with the position in which the three points are taken; and if the points be assumed to be indefinitely near to each other, the curves will osculate in their vicinity. Thus, in general, any curve of mortality may be considered as accurately represented by Gompertz's formula, if the symbols k, g, q, instead of being constants, are supposed to denote functions of the age x, and to appear as variable parameters that usually undergo but gradual changes in value." I have now to make the further suggestion that any curve of mortality may be accurately represented by Gompertz's formula if any one of those symbols is alone supposed to be a function of x and to appear as a variable parameter, the other two symbols being retained as constants throughout the curve. The conspicuous advantage derivable from Gompertz's formula when thus applied will be the mathematical fact that such parameter will in general not be subject to any capricious changes in value, and will therefore be convenient to deal with as a sort of key to the table of mortality.

Take the adjusted Experience Table as an example and let q be the variable parameter. Also, consulting the values contained in the table, vol. x., page 124, let the assumed constants be $\log k = 5{\cdot}20000$, $\log g = -0{\cdot}20000$; so that the formula becomes

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* It will be observed that Mr. Jellicoe, in his investigation just referred to, has not failed, with the exercise of tact and discrimination, to effect a practical remedy for the defective continuity observable at these points.

$$\log l_x = 5{\cdot}20000 - \tfrac{1}{5}q^x.$$

Then the values of the subsidiary number, $\log q$, may be found as follows:—

Age x	$\log l =$ $\log k + \log g.q^x$	$\log g.q^x$	q^x	$x \log q$	$\log q$
10	5·0000000	−0·2000000	1·0000000	0·0000000	·0000000
15	4·9851389	0·2148611	1·0743055	0·0311278	·0020752
20	4·9697327	0·2302673	1·1513365	0·0612022	·0030601
25	4·9534456	0·2465544	1·2327720	0·0908827	·0036353
30	4·9359705	0·2640295	1·3201475	0·1206224	·0040207
35	4·9168801	0·2831199	1·4155995	0·1509404	·0043126
40	4·8957153	0·3042847	1·5214235	0·1822501	·0045563
45	4·8717772	0·3282228	1·6411140	0·2151387	·0047809
50	4·8420910	0·3579090	1·7895450	0·2527426	·0050549
55	4·8025617	0·3974383	1·9871915	0·2982397	·0054225
60	4·7479786	0·4520214	2·2601070	0·3541290	·0059021
65	4·6698188	0·5301812	2·6509060	0·4233943	·0065138
70	4·5543316	0·6456684	3·2283420	0·5089795	·0072711
75	4·3820170	0·8179830	4·0899150	0·6117143	·0081562
80	4·1235250	1·0764750	5·3823750	0·7309740	·0091372
85	3·7337588	1·4662412	7·3312060	0·8651755	·0101785
90	3·1202448	2·0797552	10·3987760	1·0169822	·0112998
95	1·9493900	−3·2506100	16·2530500	1·2109348	·0127467
100	−∞	−∞	∞	∞	∞

The resulting values of $\log q$, contained in the last column, judging by their progression, appear to be somewhat tractable.

Moreover, the principle originated by Gompertz also suggests another practical method of simplifying a discussion of the unadjusted results of a mortality table. Let $P_x = \log(\log l_x - \log l_{x+t})$ denote the logarithm of the logarithm of the reciprocal of the probability of a life, age x, surviving t years. Then since, according to Gompertz's law, a series of such values, viz. P_x, P_{x+t}, P_{x+2t}, &c., should be in arithmetical progression, we may at least infer that they are approximately so, and conclude that, in consequence of the smallness of their second differences, they must be a convenient set of numbers to have recourse to for the purposes of adjustment, so as to obtain a suitable continuity of progression in the final table. But, after the unexpected length to which this paper has already extended, I must forego any detailed discussion of what might be accomplished by the expedient here suggested, which would seem to have some promise of being not only interesting but of real practical utility.

CORRESPONDENCE.

MR. STEPHENSON'S LETTER ON THE VALUE OF OPTIONS.

To the Editor.

SIR,—I venture to think that Mr. Stephenson has not correctly defined the weak point in the Government Annuity Table, commented on by him in the April number of the *Journal*.

The benefits provided for by the table, in return for a single premium to be paid, are* (1) an assurance for n years of the amount of the premium, payable should the assured die within that period; and (2), an annuity to be entered on at the expiry of n years, provided the assured is then alive.

It appears further to be a condition of the transaction, that, in the event of the assured wishing to surrender his policy before entering on his annuity, the surrender value to be allowed him is to be limited to the amount of payment first made. Practically we might assume that the advantage to the Government by such a return is no greater than to form a legitimate profit on a surrender policy; at any rate, in the absence of data as to the proportion of policies thus given up, it is impossible to form any estimate of the probable gain from them.

It follows, therefore, that, for payment of a single premium of £1, the deferred annuity to be granted, allowing for the short assurance, is, by commutation table formula,

$$\frac{D_x-(M_x-M_{x+n})}{N_{x+n}};$$

and that the single premium for an annuity of £1, expressed in terms of Column N, for comparison with Mr. Stephenson's formula for it, is

$$\frac{N_{x+n}}{(1-v)(N_{x-1}-N_{x+n-1})+D_{x+n}}.$$

After forming a correct estimate of the deferred annuity *benefit*, Mr. Stephenson proposes to deal with the *payment* as of one where the Government are to be in the receipt of the interest of it only for n years, receiving the principal at the end of that term should the assured be then alive, and thus providing for the contingency of his death.

The present value of such a payment, where the single premium is nominally £1, is (r being the interest)

$$\frac{r(N_x-N_{x+n})+D_{x+n}}{D_x};$$

but this does not truly represent the Government scheme of payment, inasmuch as if the assured die, the repayment to his representatives of £1 being made only at the end of the year of death, a year's interest has accrued thereon, for the disposal of which Mr. Stephenson's formula has not provided, causing the difference between it and the one first above written.

Of course there is no consideration of surrender value enters into the question.

I am, Sir,
Your most obedient servant,

Aberdeen, 5th June, 1865. H. AMBROSE SMITH.

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* I have not seen the table, but adopt Mr. Stephenson's description of it.

ON THE VALUE OF OPTIONS IN CERTAIN CONTRACTS.

To the Editor.

SIR,—Mr. Stephenson's problem on "Options" continuing to excite a good deal of interest, I shall be obliged if you will allow me space for a somewhat full discussion of it.

As stated by Mr. Stephenson the problem is:—To find the single premium for an annuity [of £1] during the remainder of a life (x) after n years, with the condition that the premium is "returnable," without interest, on death or at the option of the purchaser, at any time prior to the commencement of the annuity.

To put this into shape for solution, Mr. S. directs us to consider the single premium, P, as a capital producing in interest Pr per annum; and that this interest, treated as a premium payable at the end of the year, will purchase a certain amount of annuity, to be entered upon in n years. Also, that the capital P being then invested in the purchase of a further amount of annuity, since the whole of the annuity purchased is to be £1, we shall have an equation that will enable us to determine P.

I remark in passing, that the portion of the benefit during the n years here contemplated is slightly different from that in the problem. According to the latter the return is to be P, at the instant of death or withdrawal, the amount of which at the end of the year will be $P(1+r)^{\frac{1}{2}}$. By the other arrangement the amount in the hands of (x) or his representatives, at the same period, would be $P(1+r)$, since, no premium being payable during the year of death or withdrawal, the parties entitled will be in possession of both P and the interest on it for that year. This is a small matter, however.

Mr. Stephenson's solution of the problem, as here modified, is equivalent to the following:—

The present value of the premium Pr, payable at the end of each year for n years, is

$$\frac{Pr(N_x-N_{x+n})}{D_x};$$

and if a be the annuity that this will purchase, we shall have

$$Pr(N_x-N_{x+n})=aN_{x+n}; \quad \therefore a=\frac{Pr(N_x-N_{x+n})}{N_{x+n}}.$$

Also, if a' be the annuity that P will purchase at age $x+n$,

$$P=\frac{a'N_{x+n}}{D_{x+n}}; \quad \therefore a'=\frac{PD_{x+n}}{N_{x+n}}.$$

Hence, by condition,

$$\frac{PD_{x+n}+Pr(N_x-N_{x+n})}{N_{x+n}}=a'+a=1;$$

which gives

$$P=\frac{N_{x+n}}{D_{x+n}+r(N_x-N_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now there is a gross paralogism in this process: the very first step is vicious. $Pr(N_x-N_{x+n})\div D_x$ is the value of the premium Pr when it is determinable (during the first n years) by the death of (x) only; but it is not the value of the same premium when it is determinable *also* at the pleasure or the caprice of (x) by the withdrawal of P.* It is as representing this last-named value that it is employed in the foregoing process, which is therefore vitiated throughout. And the value of P here found is that corresponding to the legitimate signification of the expression referred to; and consequently no provision is made in it for the option of withdrawal.

I shall show this otherwise. Mr. Stephenson's benefit, *divested of the option clause*, and supposing, as above, that the amount of the return at the end of the year of death is $P(1+r)$, consists of, first, an annuity of £1, deferred n years, whose present value is $\frac{N_{x+n}}{D_x}$; and secondly, a temporary assurance for n years of $P(1+r)$, whose present value is $\frac{P(1+r)(M_x-M_{x+n})}{D_x}$. Hence

$$PD_x=N_{x+n}+P(1+r)(M_x-M_{x+n});$$

solution of which gives

$$P=\frac{N_{x+n}}{D_x-(1+r)(M_x-M_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now (1) and (2) are identical in value: each of them gives, for $x=50$ and $n=10$ (Carlisle, 4 per cent.), $P=6{\cdot}30295$. And they will become identical in symbols also if in (2) we substitute for M_x and M_{x+n} their values as given by the relation $M_x=D_x-(1-v)N_{x-1}=D_x-(1-v)(D_x+N_x)$. It is thus abundantly shown (although it looks like trifling to offer formal demonstration of so plain a matter) that a method which ignores the probability of withdrawal does not, and cannot, assign the true value of a benefit which depends partly on that probability.

If the value of P when returnable *at the moment of death* is required (and still, of course, without the option of withdrawal), we have merely to write $(1+r)^{\frac{1}{2}}$ for $1+r$ in (2), which thus becomes

$$P=\frac{N_{x+n}}{D_x-(1+r)^{\frac{1}{2}}(M_x-M_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

And this gives, for the same values of x and n as before, $P=6{\cdot}2828$. Or, finally, by omitting from (3) the factor $(1+r)^{\frac{1}{2}}$ we get for the value of P, when returnable at the end of the year of death, 6·26313.

The benefit described in the problem, however, is altogether delusive. It seems to have been devised with the object of enabling the purchaser to made a provision for advancing years, while retaining at the same time a certain amount of control over his capital. If so the object can be attained more simply and more efficaciously otherwise. Let the intending purchaser retain his capital in his own hands, and, when his necessities so require, let him invest its improved amount in an immediate annuity, and he will find that he has done just as well as he would have done by the arrangement in the problem, while he has retained his control over both the capital and its interest.

To prove this:—Let the capital as before be P. Its amount in n years will be $P(1+r)^n$; and, if this sum be invested in an annuity of £1, we shall have

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* This is very clearly pointed out by Mr. Younger at p. 55 *ante*.

$$P(1+r)^n=\frac{N_{x+n}}{D_{x+n}};$$

whence, $$P=\frac{N_{x+n}}{(1+r)^n D_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

For $x=50$ and $x+n=60$, (1) and (2) give, as we have seen, $P=6{\cdot}30295$; while (4) gives $P=6{\cdot}52820$. This slight increase in the value of P is the equivalent for the advantage of being relieved from the risk of losing the interest in the event of death during the n years.

For the proper solution of Mr. Stephenson's problem it is (or ought to be) obvious, that we require to know the law that governs the withdrawals as well as that which governs the mortality, the influence upon the result exercised by the one of these elements being entirely analogous to that exercised by the other. In attempting a solution of the problem therefore, as a matter of scientific interest, it is necessary to assume a law of withdrawal. This is done by Mr. Younger in the solution given by him in the last number of the *Journal*, pp. 55 to 58; and in that which I am about to offer I shall employ the same law. Of Mr. Younger's solution I shall now only say, that I consider the analysis he employs as of a much more refined character than the circumstances of the case require.

If λ_x denote the number of policies at age x, the numbers of these remaining in force at the end of 1, 2, . . . $t-1$ years, will be denoted by λ_{x+1}, λ_{x+2}, . . . λ_{x+t-1}, respectively. Now, let the law of withdrawal be, that the number of withdrawals in each year is the fraction k of the number of policies in force at the beginning of the year; so shall the withdrawals during the 1st, 2nd . . . tth years be severally denoted by $k\lambda_x$, $k\lambda_{x+1}$ . . . $k\lambda_{x+t-1}$. Let also the withdrawals of each year, and in like manner the deaths, be uniformly distributed over the year.

The decrement of the tth year, if there were no withdrawals, would be the product of λ_{x+t-1} by the probability of $(x+t-1)$ dying in a year. The withdrawals during the year are $k\lambda_{x+t-1}$, at equal intervals therein, which is equivalent, as regards the operation of mortality, to the whole taking place in the middle of the year; and this again is equivalent, in the same sense, to one-half taking place at the commencement of the year, and the other half at the end. Hence the number exposed to the risk of mortality during the whole of the tth year will be denoted by $\lambda_{x+t-1}-\frac{1}{2}k\lambda_{x+t-1}$, or $\lambda_{x+t-1}(1-\frac{1}{2}k)$, which gives for the portion of the decrement due to mortality $\lambda_{x+t-1}(1-p_{x+t-1})(1-\frac{1}{2}k)$; and adding to this the portion due to withdrawals, $k\lambda_{x+t-1}$, we get for the total decrement of the tth year,

$$\lambda_{x+t-1}\{(1-p_{x+t-1})(1-\tfrac{1}{2}k)+k\}.$$

Hence,

$$\begin{aligned}\lambda_{x+t}&=\lambda_{x+t-1}-\lambda_{x+t-1}\{(1-p_{x+t-1})(1-\tfrac{1}{2}k)+k\}\\&=\lambda_{x+t-1}\{(1-\tfrac{1}{2}k)p_{x+t-1}-\tfrac{1}{2}k\}.\end{aligned}$$

And if we denote by ϖ_{x+t-1} the probability that a policy in force at the beginning of the tth year will not be extinguished that year, but will remain in force at the end, we shall have

$$\varpi_{x+t-1}=\frac{\lambda_{x+t}}{\lambda_{x+t-1}}=(1-\tfrac{1}{2}k)p_{x+t-1}-\tfrac{1}{2}k.$$

From the nature of this function we shall obviously have

$$\lambda_{x+t}=\lambda_x.\varpi_x.\varpi_{x+1}\ .\ .\ .\ \varpi_{x+t-1};$$

and hence, with any assumed value of λ_x as a radix, we shall be able to form with facility a table of the number of policies in force at the end of each successive year, up to age $x+n$, at which the power of withdrawal ceases. The number then in force being subject to diminution by mortality only, the portion of the table from this point will be identical with the corresponding portion of the mortality table if the radix λ_x be so assumed that λ_{x+n} shall equal l_{x+n}. This is a matter easily arranged.

Mr. Younger, in his example, employs the Carlisle mortality, and assumes $k=\frac{1}{20}=\cdot 05$. I have formed the following table on the same hypotheses, to facilitate the comparison of my result with his. It coincides with the Carlisle table at 60, the age at which, in Mr. Younger's example, the power of withdrawal ceases and the annuity is entered upon.

Policy Table.

x.	λ_x.	Deaths and Withdrawals.	Deaths.	Withdrawals.
50	7380·54	465·58	96·56	369·02
51	6914·96	442·10	96·36	345·74
52	6472·86	419·56	95·93	323·63
53	6053·30	397·97	95·31	302·66
54	5655·33	375·94	93·17	282·77
55	5279·39	356·22	92·26	263·96
56	4923·17	337·35	91·20	246·15
57	4585·82	322·72	93·43	229·29
58	4263·10	313·76	100·61	213·15
59	3949·34	306·34	108·87	197·47
60	3643·00	122·00	122·00	

This table, after what precedes, stands in need of no explanation. The last two columns, which show the deaths and withdrawals separately, have been added to afford the means of testing the accordance of the table with the laws used in its formation.

The most convenient way of adapting for use the data with which we are now furnished, will be to form by means of them a commutation table. This I have done as follows:—

Commutation Table.

x.	D'_x.	N'_x.	M'_x.
50	1038·536	9385·130	637·625
51	935·600	8449·530	574·632
52	842·097	7607·433	517·116
53	757·225	6850·208	464·632
54	680·233	6169·975	416·763
55	610·591	5559·384	373·284
56	547·492	5011·892	333·670
57	490·362	4521·530	297·597
58	438·321	4083·209	264·415
59	390·443	3692·766	233·396
60	346·305	3346·461	204·275

The rate of interest involved is 4 per cent., and the table coincides with Mr. Jones's Table XIII. (p. 295), at age 60. To distinguish values affected by withdrawal, I accent the letters designating the columns in which they are found. It is, of course, indifferent whether the values belonging to age 60 be accented or not.

The table is adapted to the treatment of any case in which x is not less than 50 and the age at which the annuity is to be entered upon is 60. In Mr. Younger's example x is 50; and the formula of solution is evidently (3), which for these values of x and $x+n$ becomes

$$P=\frac{N'_{60}}{D'_{50}-(1\cdot 04)^{\frac{1}{2}}(M'_{50}-M'_{60})}.$$

The arithmetical operation is as follows:—

M'_{50}	637·625	
M'_{60}	204·275	
	433·350	log 2·636839
	$(1\cdot 04)^{\frac{1}{2}}$	„ 0·008517
	441·933	„ 2·645356
D'_{50}	1038·536	
	596·603	„ 2·775685
N'_{60}	3346·46	„ 3·524586
P	5·60920	„ 0·748901

The value here found for P, 5·60920, is less than that given by (3), 6·2828, which is the value when there are to be no withdrawals. Mr. Younger, however, finds for it 9·4157, which is no less than 50 per cent. greater than that given by (3), as above; and he intimates his opinion that this is what ought to be. It appears to me likely that the opinion thus indicated is merely an inference from his numerical result, since a very little consideration suffices to show that the value here ought to be less than that given by (3), and not greater. Every extinction of a policy previous to the age at which the annuity is to be entered upon (and for some years after) is a decision of the risk on the policy extinguished, in favour of the Office; and hence the introduction of a condition that will increase the number of such extinctions requires and necessitates, for the preservation of the equitable character of the transaction, a diminution and not an increase of the premium. The power to withdraw is thus not a *privilege* accorded to the policy-holder, and for which he ought to be made to pay, but a *concession* made by him, in consenting to abandon his claim to the annuity in consideration of receiving back only the premium he has paid. This being so, then, it follows that Mr. Younger must have fallen into error in his solution of the problem. That he has done so I proceed to show.

We have seen that the expression (3), which, if applied by means of the ordinary table, gives the value of P when there are to be no withdrawals, gives the value also, if applied by means of *a suitable table* when there are to be withdrawals. The expression is

$$P=\frac{N_{x+n}}{D_x-(1+r)^{\frac{1}{2}}M_{x|n}};$$

and it becomes, on division of numerator and denominator by D_x,

$$P = \frac{1}{1 - \frac{(1+r)^{\frac{1}{2}} M_{x|n}}{D_x}} \cdot \frac{N_{x+n}}{D_x};$$

or, writing Q for $\frac{(1+r)^{\frac{1}{2}} M_{x|n}}{D_x}$, which is the value of a return of £1,

$$P = \frac{1}{1-Q} \cdot \frac{N_{x+n}}{D_x}.$$

This is Mr. Younger's expression, which, being thus identical with mine, ought, if properly applied—if the proper values of the elements composing it be made use of—to give the same numerical result. Mr. Younger, by an analytical process, which I think I am warranted in calling unnecessarily refined, determines an expression for Q which enables him to assign ·42556 as its value in the case in hand, which value differs but little from that given by my table—namely, ·425534. It is in the remaining element of the expression then—the annuity value—that the principal source of the discrepancy must be sought. Accordingly we find that Mr. Younger here uses the ordinary deferred annuity value; and in so doing—ignoring one of the contingencies on which, during the first 10 years, the value of the annuity depends—he, singular to say, commits an error precisely similar in character to that which he points out as vitiating Mr. Stephenson's investigation. The effect is, that the value thus assigned to P is one that will provide an annuity not only for those who neither die nor withdraw during the 10 years, but also for such of the latter class as shall survive the term!

Using the annuity value derived from my table, and Mr. Younger's value of Q (in which there is probably some arithmetical error), his formula gives for P, 5·60944.* Using also my value of Q, it gives, of course, 5·60920.

I must apologise for having occupied so much space. I hope, however, it may be found that something has been done towards the elucidation of the various points of interest that have arisen.

I am, Sir,
Yours most obediently,
P. GRAY.

London, 11*th May*, 1866.

ON MR. YOUNGER'S LETTER, AND ON THE GENERAL SOLUTION OF PROBLEMS INVOLVING DISTINCT CONTINGENCIES.

To the Editor.

Sir,—After the non-success of my endeavour to convince Mr. Stephenson of the failure of his attempt to solve a new problem in life contingencies, I declared my intention of retiring from the contest; as I felt satisfied that a continuance of the discussion with an opponent who (in the face of the evidence I had adduced) still adhered to his notion that he had succeeded in determining the *value* of the "option of withdrawal," was not likely to lead to any useful result. The subject, however, having been taken up by Mr. Younger, in an able letter published in your last Number, wherein that gentleman (after endeavouring to point out the source of Mr. Stephenson's singular error) proceeds to reduce the problem within the bounds of possibility—so far at least as regards its *theoretical* solution—by assuming the existence of a definite law of withdrawal, I now venture to request the favour of some further space for the discussion of the subject on its new footing.

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* See Mr. Younger's letter, p. 118.—Ed. *J. I. A.*

Whatever opinion may be entertained of the practical value of the inquiry (on which point, before concluding, I shall have a few words to say), there can be no question that the investigation of the problem, on the basis upon which Mr. Y. has placed it, is a perfectly legitimate subject for the pages of the *Journal*, even if it be regarded merely in the light of a mathematical exercise. I propose, therefore, in the first place, to show wherein Mr. Younger has erred in the solution which he has given; and in the second, to point out what appears to me the true mode of dealing generally with problems of this and of a cognate character.

That Mr. Younger's solution *is* erroneous is apparent from the result of his numerical example. For he finds that, upon the hypothesis of each year's withdrawals being one-twentieth of the number existing at the beginning of the year, the premium for an annuity on a life aged 50, deferred 10 years, receives an addition, consequent upon the introduction of the option of withdrawal, of no less than 50 per cent.! Now I am prepared to maintain, that, in consequence of the "value" of a policy of this description being (as I had occasion to point out in a former letter) always *in excess* of the amount of the premium paid, it must inevitably follow that the limitation of the sum to be received on withdrawal to the amount of the premium *only*, must, as a matter of calculation, have the effect of *reducing* the premium instead of *raising* it. And examining Mr. Younger's equation of condition, $Q.P_x + \frac{N_{x+n}}{D_x} = P_x$, where Q denotes the value of £1 to be paid at death or withdrawal, we may see at once that it contains a very material error, inasmuch as it assumes that the value of the *annuity* is affected by the risk of *death* only, in the period during which it is deferred. The equation, in fact, expresses the condition necessary for determining the value of a deferred annuity with the return of the premium in the event of death, and also with the option of receiving a *gratuity* of P_x (unconditionally) any time before the annuity commences—assuming, of course, that the number who in each year will avail themselves of this option will be limited in accordance with the supposed "law of withdrawal."

But independently of the oversight here adverted to, I find fault with Mr. Younger's solution as being by no means the simplest which the problem admits of. For if we consider the actual effect of the introduction of the contingency of withdrawal, we shall see, first, that its operation is confined to the period intervening between the date of the contract and the commencement of the annuity; and, secondly, that during the period in question its effect is identical with that of the risk of *death*. We have, therefore, in dealing with the problem, merely to form a table of the numbers remaining from year to year in a body or community subject to the combined action of death and withdrawal from age x (supposed to be the age at entry) to age $x+n$, at which the annuity is to be entered upon; and then to continue it from the latter age upon the supposition that the members then remaining are affected by the risk of death only. The usual D and N columns being formed from a table so constructed, the ordinary formula for a deferred annuity with the return of the premium at death, viz., $\frac{N_{x+n}}{(N_{x-1}-N_{x+n-1})(1-v)+D_{x+n}}$, should give the value of the proposed benefit.

Seeing, then, that the whole difficulty of the problem lies in the construction of a table of the values of λ_x, or the numbers remaining, at each successive year of age, among a body of members subject to death and withdrawal, I shall confine myself (so far as regards this part of my subject) to the consideration of the best means of effecting this object. And first I observe that, in the numerical solution of his problem, Mr. Younger assumes that the number withdrawing in each year bears a constant proportion to the number entering upon the year—an assumption which involves the supposition that the inclination or disposition to withdraw increases with the risk of death, for otherwise the proportion of yearly withdrawals would become less as the average quantity of existence enjoyed during each year diminishes. I shall, on the other hand, assume the law of withdrawal to be entirely independent of the law of mortality—a supposition which, besides being somewhat more consistent with reason, will, I think, be still more fully justified by the simplicity of the results to which it will lead us.

With the view of generalizing the investigation, and of adapting it to the assumption last mentioned, let us (in the first instance) discard the ideas of death and withdrawal, and suppose merely that the body or community under observation, is subject to two independent contingencies, neither of which can occur more than once to the same individual, but the happening of either of which will in no way affect the occurrence of the other. Let p denote the probability that, in reference to any particular individual, a year will elapse without the occurrence of the first contingency (which we will designate by c_1), and p' the same probability as regards c_2, the second contingency. We shall then have the following five possible cases:—

Case.	Event.	The probability of which is
1	That neither c_1 nor c_2 happens.	pp'.
2	That c_1 happens and c_2 not.	$(1-p)p'$.
3	That c_2 happens and c_1 not.	$(1-p')p$.
4	That c_1 and c_2 both happen: c_1 first.	$\frac{1}{2}(1-p)(1-p')$.[*]
5	„ „ c_2 first.	$\frac{1}{2}(1-p)(1-p')$.[*]

By combining these elementary cases, we obtain the following three, which are all we shall require for the purposes to which our conclusions will be applied:—

Cases.	Event.	The probability of which is
1	That neither c_1 nor c_2 happens.	pp'.
2 and 4	That c_1 happens, c_2 not having previously occurred.	$\frac{1}{2}(1-p)(1+p')$.
3 and 5	That c_2 happens, c_1 not having previously occurred.	$\frac{1}{2}(1-p')(1+p)$.

---

[*] According to the hypothesis usually resorted to in life contingencies, viz., that if an event is certain to happen in a given year, it is as likely to occur in any one part of that year as in any other part.

Let the contingency c_1 *now* represent death, and c_2 withdrawal; let the age of the individual be x, and let p_x and p'_x denote the respective probabilities of surviving, and of non-withdrawal, for one year—the former determined upon the supposition that there are no withdrawals, and the latter upon the supposition that there are no deaths. We shall then have, by substitution in the last table, for the probability that the individual

(1) remains to the end of the year (*i.e.*, that he
neither dies nor withdraws) . . . $p_x . p'_x$;
(2) disappears by death within the year . . $\frac{1}{2}(1-p_x)(1+p'_x)$;
(3) disappears by withdrawal within the year . $\frac{1}{2}(1-p'_x)(1+p_x)$.

These expressions afford a very easy mode of constructing a table of (λ_x), the numbers remaining at the expiration of each successive year of age, as well as of the deaths and withdrawals at every age; and it will be seen that the formulæ are precisely the same as would be used for calculating the probabilities of joint existence and of survivorship between two lives of equal ages, but subject to different laws of mortality. It is also evident that with the functions in question tabulated for every age we shall be furnished with every requisite for calculating directly the values of all contingencies depending upon the probabilities of death and withdrawal, whether separately or in combination with each other.

Let us now suppose that the disposition to withdraw does not vary, and put $p'_x=k'$ a constant quantity. We then have $\frac{\lambda_{x+1}}{\lambda_x}=\frac{l_{x+1}}{l_x}\cdot k'$, and generally $\frac{\lambda_{x+n}}{\lambda_x}=\frac{l_{x+n}}{l_x}k'^n$, a condition which is satisfied by making $\lambda_x=l_x k'^x$.

I shall now apply the results we have obtained to the consideration of a question of some practical importance, viz., the determination of the rates of premium to be charged for assurances on the lives of persons exposed to risk of death from extraneous causes—that is, from causes independent of the ordinary law of mortality. The usual (perhaps I might almost say the invariable) way of dealing with these cases is to charge a fixed extra premium, irrespective of the age of the party or of the nature of the assurance, which extra premium is discontinued when the extra risk ceases to be incurred. Now the foregoing investigation shows that we have already at our command the means of determining the proper extra premium for every case in the ordinary D and N columns calculated at different rates of interest, for we have seen that the constant extra risk (represented by k') affects these functions precisely in the same way as the interest of money.

I shall not attempt to determine the difference between the *usual* mode and the *correct* mode of dealing with these risks in all the various forms in which the cases may present themselves, but shall confine myself to the following specimens of two of the more usual ones, viz., single life assurances for one year and for the whole term of life. The formula for an assurance for one year on a life exposed to extra risk will be $v\left(1-\frac{\lambda_{x+1}}{\lambda_x}\right)$ $=v\left(1-\frac{l_{x+1}}{l_x}k'\right)=v-\frac{l_{x+1}}{l_x}k'v=v-\frac{l_{x+1}}{l_x}v'$ (putting $k'v=v'$). If we denote $l_x v'^x$ by D'_x, this becomes $v-\frac{D'_{x+1}}{D'_x}$. Again, representing

$(D'_{x+1}+D'_{x+2}+\ldots)\div D'_x$ by a'_x, the formula for the annual premium for a whole life assurance becomes $\frac{1}{1+a'_x}-(1-v)$. By these formulæ the values in the second division of the following table have been computed, vk' or v' being taken$=1{\cdot}06^{-1}$.

Carlisle Mortality—Interest 4 per Cent.
Annual Premiums for an Assurance of £100, with a Loading of 30 per Cent.

Age.	I. Without extra Risk.		II. With extra Risk.		III. Difference.		Age.
	One Year.	Whole Life.	One Year.	Whole Life.	One Year.	Whole Life.	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	
20	0 17 8	1 14 3	3 4 5	3 15 3	2 6 9	2 1 0	20
40	1 12 7	3 1 9	3 19 1	5 0 0	2 6 6	1 18 3	40
60	4 3 10	7 3 10	6 9 4	8 19 6	2 5 6	1 15 8	60

We see by this table, that when the extra risk is not influenced by the age of the individual, the practice of providing for it by a fixed addition to the annual premium is a very clumsy and inefficient one. In the case of "whole life" assurances the extra premium required is greater (and not insignificantly so) at the lower than at the higher ages; while it is materially greater in "term" than in "whole life" assurances.

A few words now, in conclusion, in reference to the practical utility of Mr. Younger's assumed "law of withdrawal." We have seen that, in the particular case to which it has been applied, the effect of its introduction into the calculation must be to *reduce* the premium; and, therefore, I think nothing more need be said against its application to problems of this class. Nor, in my opinion, would it be a whit more reasonable if applied to cases where the "value of the policy" is less than the amount of premium paid upon it (which is always the case where the interest on the premium paid is insufficient to cover the risk incurred), and where, consequently, the option of withdrawal would have a positive value. For in that case those policy-holders who refrain from exercising the option (which they are supposed to have paid for) are at a disadvantage as compared with those who avail themselves of it; and it would in reality be to the interest of the former to withdraw for the purpose of effecting new assurances with the premiums refunded to them. To illustrate this point, let us suppose that an ordinary life assurance is effected (by an annual premium) with the condition that, *during the first year only*, the option of withdrawal shall be allowed. Let us suppose, in the first instance, that no extra charge is made for this option; then, starting with the assumption that the value of the policy at the end of the first year is less than the premium paid, or that

$$1-\frac{1+a_{x+1}}{1+a_x}<\frac{1}{1+a_x}-(1-v); \quad [1]$$

that is,

$$(1+a_{x+1})(\pi_{x+1}-\pi_x)<\pi_x, \quad [2]$$

(π_x denoting the pure annual premium), we see from [2] that this *inequation* holds good if for *pure* we substitute *loaded* premiums, whether the loading be in the shape of a constant percentage, a constant addition, or a combination of the two. For in the first case we should have

$$(1+a_{x+1})(c\pi_{x+1}-c\pi_x) < c\pi_x, \qquad [3]$$

which evidently follows from [2] In the second

$$(1+a_{x+1})(\overline{\pi_{x+1}+c}-\overline{\pi_x+c}) < \overline{\pi_x+c}, \qquad [4]$$

$$\text{or } (1+a_{x+1})(\pi_{x+1}-\pi_x) < \overline{\pi_x+c},$$

which also follows evidently from [2]. And the third case is already proved, as it follows from (3) and (4) by precisely the same processes.

It appears, then, that whatever be the nature of the loading, $\pi'_x > (1+a_{x+1})(\pi'_{n+1}-\pi'_x)$, where π'_x denotes the loaded premium, follows from $\frac{1}{1+a_x}-(1-v) > 1-\frac{1+a_{x+1}}{1+a_x}$; and, indeed, a little attention to the preceding process will show that the inequality is *increased* by the loading. Now $(1+a_{x+1})(\pi'_{x+1}-\pi'_x)$ is the equivalent which an Office would require to be paid for granting a new assurance at age $x+1$ at the rate of premium for age x; and as this has been shown to be less than π'_x, it appears that when no extra charge is made for the option of withdrawal, the policy-holder would derive an advantage by withdrawing and effecting a new assurance in the way indicated. But if π'_x is greater than $(1+a_{x+1})(\pi'_{x+1}-\pi'_x)$, still more so is π'_x+e, e being any positive quantity. So that the imposition of an extra charge (e), to cover the risk of withdrawal, merely has the effect of increasing the inducement to withdraw—viz., the saving which the policy-holder would effect by availing himself of the option allowed him for the purpose of effecting a new assurance.

For convenience of illustration I have restricted the option of withdrawal to one year; but I submit that my example is sufficient to show that the system—if a practical application of it should ever be attempted—would rest on a very sandy foundation. As observed in my letter which appeared in July last, the option of withdrawal (being a contingency depending upon the *will* of the individual) is not a benefit susceptible of valuation, and it can be safely allowed only where the value of the policy is equal to or greater than the premium paid—or, in other words, either where there has been no risk incurred, or where the *interest* on the premium paid is alone a sufficient compensation for the risk. In such cases no extra charge is necessary; but whether the option should not, in certain instances, be subject to a restriction as regards the health of the policyholder, is a question altogether foreign to the points I have touched upon.

I remain, Sir,
Your very obedient servant,

London, 24th May, 1866. W. M. MAKEHAM.

ON THE TABLES OF DEFERRED ANNUITIES AS PUBLISHED BY THE GOVERNMENT.

To the Editor.

DEAR SIR,—Since I wrote to you last year on the question of options, &c., two letters have appeared in your *Journal* on the same subject; one by Mr. Makeham and the other by Mr. Younger—both writers differing much from me, but more from each other—the latter offering a mathematical solution of the problem, *purporting* to show that "a very moderate supposition of the probability of withdrawal increases the premium more than 50 per cent.;" a problem which the former has declared to be "impossible," as the value of such policies always exceeds the premiums paid upon them. Before, however, more particularly noticing the letters of these gentlemen, I beg to submit to their serious consideration the following hypothetical case for the opinion of any actuary.

A. B., aged x years, has paid the principal sum of £1 to C. D., on the trusts following (that is to say):—

1st. To invest the interest on the above sum, year by year, in the assurance of an annuity to be continued during the remainder of the life of A. B. after n years.

2nd. At the end of the said term of n years to sink the above sum of £1 in the purchase of an immediate annuity payable during the life of the said A. B., who will then (if alive) have attained the age of $\overline{x+n}$ years.

3rd. The above trusts are subject nevertheless (as far as the principal sum is concerned) to an absolute power of revocation by the said A. B., *his executors or administrators*, at any time during the said term of n years. Should the power of revocation be exercised, all claim to interest is abandoned.

Mr. —— is requested to state what amount of annuity can be assured by the above payment of £1; and further, to point out whether the benefits secured under the above arrangement differ from those offered by the Government under their scale for assuring deferred annuities, with the return of the premium at death, or, at the option of the purchaser, at any time before the annuity commences; and, if so, *in what respect?*

I submit that the inevitable answer to the above case will be—

1st. That the interest on the sum of £1 will assure a deferred annuity of $\dfrac{i}{p_{x\overline{n}|}}=\dfrac{(\mathrm{N}_x-\mathrm{N}_{x+n})i}{\mathrm{N}_{x+n}}$.

2nd. That the principal sum of £1 will, at the age of $\overline{x+n}$, purchase an immediate annuity of $\dfrac{1}{a_{x+n}}=\dfrac{\mathrm{D}_{x+n}}{\mathrm{N}_{x+n}}$, making together a total annuity of $\dfrac{i}{p_{x\overline{n}|}}+\dfrac{1}{a_{x+n}}=\dfrac{(\mathrm{N}_x-\mathrm{N}_{x+n})i+\mathrm{D}_{x+n}}{\mathrm{N}_{x+n}}$.

3rd. That as, under the powers of revocation, A. B. can recall the principal sum of £1 at any time during the term, and as his executors or administrators can and will, under the same power of revocation, do so at his death, it necessarily follows that the benefits conferred under the trust are identical with those assured under the Government scheme, but generally at a much lower price.

This answer is the reciprocal of the formula which I gave in my first letter, which made

$$\mathrm{P}_x=\frac{1}{\dfrac{i}{p_{x\overline{n}|}}+\dfrac{1}{a_{x+n}}}=\frac{\mathrm{N}_{x+n}}{(\mathrm{N}_x-\mathrm{N}_{x+n})i+\mathrm{D}_{x+n}}.$$

But if this be not the *true* premium for the assurance of the deferred annuity, it must be either too high or too low; if too high there must be a profit, if too low there must be a loss.

Now, as the only element to be determined (as far as the option is concerned) in this "new and impossible problem" is the true value of $p_{x\overline{n}|}$, it is in this quantity alone that profit or loss can arise; and it is quite clear (a true table of mortality being assumed) that nothing can accrue on this head except possible profit from the forfeiture of interest on the withdrawal of healthy lives.

Ought this profit, contingent on the forfeiture of policies, to enter into a calculation for determining the true premium for any assurance? In my opinion, clearly not; and I should most fairly have "laid myself open" to adverse criticism, had I, when impugning the correctness of the Government tables, based my objections, even in part, on their not having taken into consideration this equivocal source of profit. But should any of your readers think otherwise, nothing is easier than to determine the value of $p_{x\overline{n}|}$ by another table in which the decrements at each age during the term are so increased that they may accord with any theory as to the number of forfeitures or withdrawals which experience may suggest, in which case, calling the new premium $p'_{x\overline{n}|}$, and substituting in the original equation, we get $P_x = \dfrac{1}{\dfrac{i}{p'_{x\overline{n}|}} + \dfrac{1}{a_{x+n}}}$, which is an exact solution of the problem under these conditions, still further reducing the value of P_x, and proving that were it not that the option renders it necessary to pay everybody at the moment of claiming, there would be no pretence for charging more than the ordinary premium for return at death only.

If we apply this formula to the illustration given in Mr. Younger's letter, in which the age of entry is 50, the term 10 years, the probability of withdrawal at the beginning of each year $=\cdot05$, and the table the Carlisle 4 per cent., we shall find that $P_x=5\cdot69$, instead of $6\cdot303$ which is the true value as deduced from the original equation.

Having said thus much in support of my solution of the problem, I will now more particularly direct your attention to that offered by Mr. Younger, whence it appears that he has arrived at the conclusion that

$$P_x = \frac{N_{x+n}}{(1-Q)D_x}.$$

A glance at the form of this equation at once tells us that P_x increases as Q increases, until, when $Q=1$, the denominator vanishes, and $P_x=\infty$. The question then arises, what is Q, and is unity a possible value?

Mr. Younger tells us that Q is the "value of the risk of death and withdrawal"; but as all claims are paid at the moment of death or withdrawal, it follows that $Q=1$ on the supposition that all the members die or withdraw during the first year. His formula thus shows that the premium for assuring an annuity which by hypothesis can never be entered upon, is a sum of money *infinitely great.* This, it is needless to say, is an extreme case, but it might easily be shown, in numerous other instances, that Mr. Younger's formula makes the value of a deferred life annuity greater than that of an immediate perpetuity.

One reason why such remarkable results are obtained is, that the expression $\frac{N_{x+n}}{D_x}$ has been introduced, which, being a function of l_{x+n}, is inconsistent with the value of Q in which l_{x+n} has become λ_n.

Little need be said in reply to Mr. Makeham's letter, as in it he says, that if the problem had been differently stated I should not have laid myself open to the exception which he has taken.

Now, really, if in Mr. Makeham's eyes the solution was unexceptionable, and his objection was to the mode of stating the problem, "*Why did he not say so at first?*" But he has suggested that even his objection to the mode of stating the problem would have been obviated had I undertaken "to show how the assurance might be effected so as to reserve to the policy-holder the control over the premium paid, instead of proposing to find the premium required for the assurance with the option of withdrawal."

Surely this is a distinction without a difference. How is it possible to show how an assurance can be effected, without finding the premium to be paid? and how can the policy-holder have the option of withdrawal without having "control over the money," and conversely?

To my mind it is always a matter of great importance to depart as little as possible from the terms in which a problem is brought before you, as by doing so a divergence of views is rendered more immediately apparent. Had I proposed it in that which I considered its simplest form, I should have done so as follows:—

What amount of deferred annuity can be secured to a person for each payment of £1, the assured reserving to himself and his representatives the "option" of withdrawing the money so paid (without interest) at any time before the annuity commences.

In this method of treating the subject the option has a more extended signification than I had given it in my previous letter, but I have always considered this the more correct way of looking at the question, as the return of the premium at death is only a *particular* case of the exercise of the option of withdrawal, and it is moreover in conformity with the solution in which the deferred and immediate annuities are alone involved.

There are several other points in Mr. Makeham's letter which are open to objection, but to enter into these would be to depart from those broader views which ought always to be advocated in journals devoted to scientific pursuits.

In conclusion I will only add, that in introducing the subject to your notice I had no idea of proposing a new theory of the value of options, but simply of demonstrating that $\frac{N_{x+n}}{(N_x-N_{x+n})i+D_{x+n}}$ was amply sufficient to secure to an assurer all the benefits proposed to be conferred under the Government tables, including the "option" of the withdrawal of the premium paid, at any time during the term, either by himself or his representatives—and I have done so.

Yours truly,

London, 14*th May*, 1866. J. W. STEPHENSON.

ON THE VALUE OF OPTIONS IN CERTAIN CONTRACTS.

To the Editor.

Sir,—The equation of condition at the end of my letter in the last Number of the *Journal* needs a correction, owing to the accidental omission of a factor. The equation should be

$$Q.P_x + \frac{\lambda_n}{l_{x+n}} . \frac{N_{x+n}}{D_x} = P_x.$$

The annuity portion of the risk is evidently $= \frac{\lambda_n}{l_x} v^n . a_{x+n}$, or $\frac{\lambda_n}{l_{x+n}} . \frac{N_{x+n}}{D_x}$, since $\frac{\lambda_n}{l_x}$ is the probability that any one of the lives will enter upon the annuity. Hence

$$P_x = \frac{\lambda_n}{l_{x+n}} . \frac{N_{x+n}}{(1-Q)D_x}.$$

Taking the example given in my letter, we have, as before, $Q = {\cdot}42556$ and $\lambda_{10} = 2170{\cdot}32$; hence $P_x = 5{\cdot}6094$.

To determine the precise effect of the withdrawals upon the amount of the premium, we must find what the premium would be if it were returnable at the moment of death and no withdrawals were allowed. In this case it is easily found that $P_x = \dfrac{N_{x+n}}{D_x - \dfrac{i}{\log_e(1+i)}(M_x - M_{x+n})} = 6{\cdot}2828.$

It therefore appears that if the withdrawals which have been assumed were certain to take place, the effect would be to *lessen* the premium, not to increase it.

I did not perceive the omission of the factor referred to until the day the *Journal* was published, otherwise the numerical example before given would have been corrected, and the paradoxical nature of its result would thus have been removed.

At whatever time a withdrawal takes place, it is obvious that the return of the original premium without interest must leave the Office a gainer, because the risk continually increases, while the premium returned merely represents the initial value of the risk. Thus the Office is set free from a liability without being called upon to pay its full equivalent value; hence the reduction which takes place in the premium.

I am, Sir,
Your obedient servant,
SAMUEL YOUNGER.

316, *Regent Street, London,*
28*th May*, 1866.

ON THE FORMULA FOR QUADRATURES.

To the Editor.

DEAR SIR,—The June number of the *Educational Times* contains a mathematical article "On approximation to a curvilinear area," by Professor De Morgan, in which he gives the results of a determination of the numerical coefficients appertaining to the formula usually employed in calculating quadratures. The coefficients, according to a method founded on the "Calculus of operations," are those of the symbolic expansion of $\{\log(1+\Delta)\}^{-1}$. They may, therefore, be found to any number of terms by working out, by long division, the reciprocal of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$&c. Or, if C_n denote the nth coefficient, it may be computed, from those which precede it, by the formula

$$C_n=\frac{C_{n-1}}{2}-\frac{C_{n-2}}{3}+\frac{C_{n-3}}{4}-\ldots\ldots\pm\frac{C_1(=1)}{n}.$$

In allusion to these arithmetical values, Professor De Morgan's interesting note concludes as follows:—

"The coefficients of $\{\log(1+\Delta)\}^{-1}$, so far as usually given, are

$$1,\ \frac{1}{2},\ -\frac{1}{12},\ \frac{1}{24},\ -\frac{19}{720},\ \frac{3}{160},\ -\frac{863}{60480}.$$

The three next, whether ever before printed I know not, are

$$\frac{275}{24192},\ -\frac{33953}{3628800},\ \frac{8183}{1036800}."$$

The object of the present communication is merely to point out that, by an investigation conducted on the principles of ordinary algebra, the whole of these coefficients had already been determined in my paper on "Summation," printed in the *Journal.* See vol. xi., page 309, where, with respect to a series of equidistant ordinates V_0, V_1, V_2, &c., it is found that "the curvilinear area bounded by V_0 and $V_n=$

$$(V_0+V_1\ldots\ldots+V_n)-\tfrac{1}{2}(V_n+V_0)-\frac{1}{12}(a'-a)-\frac{1}{24}(b'+b)$$
$$-\frac{19}{720}(c'-c)-\frac{3}{160}(d'+d)-\frac{863}{60480}(e'-e)-\frac{275}{24192}(f'+f)$$
$$-\frac{33953}{3628800}(g'-g)-\frac{8183}{1036800}(h'+h)\ \ldots\ldots\ (B)."$$

It will be perceived that this formula includes all the coefficients stated by Professor De Morgan. Moreover, the general formula (A) expressing the result of the summation of an interpolated finite series of values is carried out to the same order of differences—that is, to the eighth order. It is, however, to the integration formula (B) only that reference is needed on this occasion; and the practical inference to be drawn is, that the identity of Professor De Morgan's numerical coefficients with those previously determined by a process so entirely different may be accepted as a satisfactory proof of their accuracy.

I am, dear Sir,
Yours most truly,
W. S. B. WOOLHOUSE.

Alwyne Lodge,
8th June, 1866.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

First Ordinary Meeting. Session 1865–6.—*Monday,* 27*th November,* 1865.

The President in the Chair.

Read and confirmed the minutes of the last anniversary meeting.

Mr. A. P. Fletcher, nominated at the last ordinary meeting, was elected a Fellow of the Institute.

Mr. Makeham read a paper "On the principles to be observed in the construction of mortality tables."

Thanks having been voted to Mr. Makeham, the meeting adjourned to Monday, the 18th December, 1865.

Second Ordinary Meeting. Session 1865–6.—*Monday,* 18*th December,* 1865.

Samuel Brown, Esq., Vice-President, in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected members of the Institute, viz.:—

Fellows.

H. S. Hawkins.
Edward A. Newton, M.A.

Associates.

Richard H. Woodman.
Robert H. Nesbitt.
John J. Ridge.
James Chisholm.
John C. Hopkinson, B.A.
James Ostler.
Hubert George Rowsell.
Joseph Allen.
Arthur Smither.
Henry M. Kember.
Walter Kember.
Charles J. Wallis.
Elias H. Symons.

Mr. M. N. Adler read a "Memoir of the late Benjamin Gompertz, F.R.S., with some account of his contributions to actuarial science."

Thanks were voted to Mr. Adler, and the meeting adjourned to the 29th January, 1866.

Third Ordinary Meeting. Session 1865–6.—*Monday,* 29*th January,* 1866.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected Associates of the Institute, viz.:—

Charles Darrell.
David Evans.
James Pierce.
William G. G. Davis.
William H. Penney.
Samuel Edwards.
Wilfred A. Bowser.
Robert W. Symes, B.A.
William E. Stark.

The President announced the following as the result of the Examinations held in December, 1865, viz.:—

MATRICULATION EXAMINATION.

Sixteen candidates presented themselves for this Examination, of whom five passed in the following order of merit, viz.:—

James Pierce.
John C. Hopkinson.
Robert W. Symes.
James Duncan.
Robert H. Nesbitt.

SECOND YEAR'S EXAMINATION.

Seven candidates presented themselves for examination, of whom one withdrew without giving in his papers, and one passed, viz.:—

Henry Mountcastle.

THIRD YEAR'S EXAMINATION.

Three gentlemen presented themselves as candidates for certificates of competency, of whom two passed, viz.:—

1st. Henry W. Manly.
2nd. David A. Bumsted.

Mr. T. B. Sprague, M.A., read a paper "On the limitation of risks; being an essay towards the determination of the maximum amount of risk to be retained by a Life Assurance Company on a single contingency."

Thanks were voted to Mr. Sprague, and the meeting adjourned to Monday, the 26th of February, 1866.

Fourth Ordinary Meeting. Session 1865–6.—*Monday,* 26*th February,* 1866.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman, nominated at the last ordinary meeting, was elected an Associate of the Institute, viz.:—

George J. Row.

Mr. W. P. Pattison read a paper "On organization, and its application to Assurance Companies."

Thanks were voted to Mr. Pattison, and the meeting adjourned to Monday, the 26th March, 1866.

Fifth Ordinary Meeting. Session 1865–6.—*Monday,* 26*th March,* 1866.

W. B HODGE, Esq., Vice-President, in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected Associates of the Institute, viz.:—

Josias G. Dwight.
William J. F. Norfolk.

Mr. Peter Gray read a paper "On the construction of tables by the method of differences."

Thanks were voted to Mr. Gray, and the meeting adjourned to Monday, the 30th April, 1866.

Sixth Ordinary Meeting. Session 1865–6.—*Monday,* 30*th April,* 1866.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman, nominated at the last ordinary meeting, was elected an Associate of the Institute, viz.:—

Charles Smith.

Mr. W. S. B. Woolhouse read a paper "On the construction of tables of mortality."

Thanks were voted to Mr. Woolhouse, and the meeting adjourned to Monday, the 26th November, 1866.

The Nineteenth Annual General Meeting, Saturday, 2nd June, 1866.

ARTHUR HUTCHESON BAILEY, Vice President, in the chair.

Mr. J. Hill Williams, Honorary Secretary, read the minutes of the preceding ordinary meeting, and the following Reports and account of receipts and payments:—

"The time has again arrived when the duty devolves upon the Council of reporting to the members of the Institute as to the state of their affairs.

"The usual letter from the Secretary which accompanies this Report, and the annual account appended to it, furnish ample details as to the condition of the Institute, and its progress during the past year.

"It will be seen that the income of the year was £474. 12*s*. 3*d*., exclusive of the interest due on the sums invested, and that the outgoings amounted to £362. 13*s*. 6*d*. The property of the Institute may now be estimated at upwards of £1,300.

"The number of members has also increased during the year, there being at the present time 220 as contrasted with 203, the number on the register at the date of the last report.

"The papers read during the Session have been as follow:—

'On the principles to be observed in the construction of Mortality Tables.' By Mr. W. M. Makeham.

'A Memoir of the late Benjamin Gompertz, F.R.S., with some account of his contributions to actuarial science.' By Mr. M. N. Adler.

'On the limitation of risks, being an essay towards the determination of the maximum amount of risk to be retained by a Life Assurance Company on a single contingency.' By Mr. T. B. Sprague.

'On organization, and its application to Assurance Companies.' By Mr. W. P. Pattison.

'On the construction of tables by the method of differences.' By Mr. Peter Gray.

'On the construction of tables of mortality.' By Mr. W. S. B. Woolhouse.

"In their last report, the Council mentioned, that the Clerical and Medical Office, the London Assurance, the Pelican, and the Guardian Offices, had each completed the collection of the data required, for determining the rate of mortality amongst the lives assured with them, and expressed the hope that other Companies would follow in the same course.

"The Offices mentioned have since put the Council in possession of the valuable data thus collected, and the Council are now engaged in the preliminary classification and arrangement of them. They hope shortly to receive the experience of some other Companies, when the needful investigation of the whole will be entered upon, with a view to the accurate solution of the very interesting and important questions involved therein.

"As it is obvious that this investigation must be attended with considerable expense, the Council thought it desirable to invite contributions towards the liquidation of it, and they are glad to be able to announce that the contributions received to the date of this Report, amount to £425. 10*s*.

"For some time past the Council have had their attention directed to the desirableness of supplying information of a more definite character than the existing syllabus affords, as to the subjects to which candidates for the Institute certificate should more particularly address themselves, and Mr. Sprague, one of the Vice-Presidents, and others, have been at considerable pains to prepare, with a view to publication for the use of students, a syllabus of a supplementary or subsidiary nature, answering the proposed end. The Council have not yet been able to arrive at a determination of the matter, but they hope soon to be in a position to do so.

"They have only further to state, that a few volumes have been purchased towards the collection which is to be formed to constitute the 'Hardy Memorial,' and that further purchases will be made as occasions present themselves."

"*Institute of Actuaries*, 12, *St. James's Square*,
28th April, 1866.

"Dear Sir,—Herewith I transmit you a copy of the audited Abstract of Receipts and Payments of the Institute of Actuaries for the financial year ended 31st March, 1866, with a few remarks.

"Receipts.

"In 1863-64 the total receipts were £521. 6*s*. 6*d*., in 1864-65 they were £640. 1*s*. 11*d*., and for 1865-66 they were £474. 12*s*. 3*d*. This decrease is owing to the fact that in 1863-64 four life subscriptions were received, making £84, and in 1864-65, £150. 13*s*. was received for the Hardy Memorial.

"The total sums received from subscriptions only in the last three years are as follow:—

1863–64.	Annual subscriptions . .	£411 12 0	
	Life " . .	84 0 0	
			£495 12 0
1864–65.	Annual subscriptions . .	£415 16 0	
	Life " . .	21 0 0	
			£436 16 0
1865–66.	Annual subscriptions . .	£464 2 0	
	Life " . .	..	
			£464 2 0

"The balance in the bank on the 31st March, 1866, was £270. 15*s*. 10*d*. as compared with £159. 12*s*. 8*d*. of the previous year.

"Payments.

"The following table gives the annual subscriptions and general expenses for the last three years:—

	Annual Subscriptions.	General Expenses.	Difference.
1863–64........	£411 12 0	£399 3 6	+ £22 8 6
1864–65........	415 16 0	460 2 11	− 44 6 11
1865–66........	464 2 0	362 13 6	+ 101 8 6

"The cost of printing the *Journal* was £86. 19*s*. 6*d*., as compared with £85. 3*s*. 6*d*. of the year previous. The payments for salaries, stationery and sundry printing, ordinary meeting expenses, lighting, postage, and house and petty expenses, have slightly increased; whilst for the library, advertising examinations, and miscellaneous expenses, they have diminished.

"ASSETS.

"The investments of the Institute on the 31st March last amounted to £646. 14*s.* 3*d.*, and the balance at the bank and the petty cash balance to £271. 14*s.* 7*d.* The total available assets at the end of the year, therefore, amounted to £918. 8*s.* 10*d.*, as compared with £806. 10*s.* 1*d.* of the previous year.

"LIABILITIES.

"The following is an estimate of the liabilities of the Institute to the 30th September next:—

Two quarters' salaries	£52	10	0
Rent (half year, due Michaelmas)	37	10	0
Journals—April and July (say)	45	0	0
Stationery and sundry printing (say)	10	0	0
Miscellaneous (say)	10	0	0
	£155	0	0

"LIST OF MEMBERS.

"On the 1st April, 1865, the total number of Members on the list was 203; during the year 30 new members were elected—viz., 3 Fellows and 27 Associates; and the losses were 13, all of whom were Associates; leaving on the list, on the 31st March, 1866, 95 Fellows and 125 Associates—in all, 220 members. During the year 2 Associates have been transferred to the list of Fellows. These transfers affect the totals of the different classes of members, but do not alter the general total.

"I am, dear Sir, yours faithfully,

"FREDERICK GOVER,

"*Assistant Secretary.*

"C. JELLICOE, Esq."

INSTITUTE OF ACTUARIES.

Dr. *Abstract of Cash Account from 1st April,* 1865, *to 31st March,* 1866. **Cr.**

RECEIPTS.

	£	s.	d.	£	s.	d.
To Cash balance brought forward	159	12	8			
Petty cash " "	0	3	2			
				159	15	10
To Subscriptions for 1865–66, viz.:—						
64 Fellows, Town, at £3 3 0	201	12	0			
22 " Country, " 2 2 0	46	4	0			
81 Associates, Town, " 2 2 0	170	2	0			
24 " Country, " 1 1 0	25	4	0			
Arrears	21	0	0			
Overpayment	0	0	3			
				464	2	3
Certificate Fees				10	10	0
				£634	8	1

PAYMENTS.

	£	s.	d.	£	s.	d.
By Rent	75	0	0			
Salaries	105	0	0			
Journal	86	19	6			
Stationery and sundry Printing	41	13	9			
Ordinary Meetings	21	8	2			
Lighting	3	16	10			
Postage and Receipt Stamps	8	10	0			
Library	5	0	8			
Advertising Examinations	3	17	0			
House and Petty Expenses	6	3	8			
Miscellaneous	5	3	11			
				362	13	6
Balance in Bank, carried forward	270	15	10			
" of Petty Cash	0	18	9			
				271	14	7
				£634	8	1

	£	s.	d.
NOTE.—Cost of £700, 3 per Cent. Consols	646	14	3
Cash balance	271	14	7
Books in Library (say)	381	11	2
	£1,300	0	0

Examined and found correct, April 27th, 1866.

(Signed) B. NEWBATT,
(") G. W. BERRIDGE,
(") RALPH P. HARDY,
Auditors.

The Chairman said—It now becomes my duty to move, "That the Report of the Council, which you have just heard, together with the Abstract of Receipts and Payments, be adopted, entered on the minutes, and printed in the *Journal.*" I do not know that it is necessary for me to accompany this resolution with any observations of my own. I hope the general body of members will agree with the Council, that the Institute fairly maintains its position. (Hear, hear.) It cannot boast of being rich; it pays its way, and that is something to say in times like these; and I think its general usefulness may be considered established. (Hear, hear.) There are one or two points of an unusual character mentioned in the Report. The first is a matter we are all interested in; and that is, the endeavour we are making to collect a larger body of information as regards the mortality of assured lives than has yet been made available. I am myself sanguine the result will redound to the credit of the Institute, and that the information which will be obtained will be of great value to the profession, and through them to the public at large. (Hear, hear.) The labour required is enormous, as every gentleman here knows. I think I may say that since this Report was drawn up, we have reason to believe the materials have been considerably increased, and also—and this was matter of some anxiety to all of us—that the necessary funds for the expenses of the undertaking will be forthcoming. (Hear, hear.) The other special matter referred to in the Report is one which, having tried my hand at unsuccessfully, I know is attended with considerable difficulty, and that is an attempt to improve the syllabus for the examinations. I for one think that the maintaining of the examinations in a state of efficiency is one of the most important duties of the Institute. (Hear, hear.) But it is a matter of great difficulty to steer between Scylla and Charybdis. On the one side, we must be careful not to make the examinations too abstruse and recondite; and on the other, we must make them sufficiently difficult to require such study as shall be of value to the candidates. This matter is now in the hands of two or three of the very ablest members of the Institute, and I hope and believe they will succeed in producing something which will give assistance to the candidates and serve the purpose of the Institute itself. With these few observations, which I have made without being at all prepared, not knowing I should be called upon to exercise this position, I beg to move the adoption of the Report. (Cheers).

Mr. Reddish seconded the resolution.

Mr. Galsworthy said it was quite clear no one felt disposed to make any observations, because the Report was so extremely satisfactory. He had not been able to attend any of the meetings of the Council, when the Report had been discussed, but he was much gratified at some of the points therein mentioned. The income was satisfactory, and there was nothing to complain of in regard to the expenditure. There was one most important feature, and that was the increase in the number of the members. So long as they could pay their way, and at the same time increase their numbers, he thought they must all be gratified. (Hear, hear.) Another satisfactory fact was, that the papers read before the Institute had not only maintained the standard of previous sessions, but if anything had exceeded it. Mr. Woolhouse, Mr. Gray, Mr. Sprague, Mr. Makeham, and other gentlemen of high eminence in the profession, had brought their valuable contributions before the Institute, and it was not altogether fair that the labours of these gentlemen, and the other active members of the Council—of which he was not one—should be passed over without recognition on these occasions. (Hear, hear.)

The Report was then unanimously adopted.

The following list of office-bearers for the ensuing year, recommended by the Council, was ballotted for, and agreed to unanimously:—

President.

Charles Jellicoe.

Vice Presidents.

Samuel Brown.
William Barwick Hodge.
Robert Tucker.
W. S. B. Woolhouse.

Council.

Arthur Hutcheson Bailey.
Samuel Brown.
Charles John Bunyon, M.A.
George Cutcliffe.
Archibald Day.
Percy Matthew Dove.
William Emmens.
Edwin Henry Galsworthy.
William John Hancock.
Augustus Hendriks.
William Barwick Hodge.
Charles Jellicoe.
Charles Terrell Lewis.
Jeremiah Lodge, B.A.
Henry Marshal.
Frank McGedy.
James Meikle.
John Messent.
Edward A. Newton, M.A.
William P. Pattison.
Arthur Pearson.
Henry William Porter.
John Reddish.
Henry Ambrose Smith.
William Spens.
Thomas Bond Sprague, M.A.
John Stott.
Robert Tucker.
John Hill Williams.
W. S. B. Woolhouse.

Treasurer.

George Cutcliffe.

Honorary Secretaries.

Archibald Day.
John Hill Williams.

Mr. Lodge said that with the permission of the chairman he would say a few words with reference to the list just read, from which he missed the name of their old and valued friend—valued by everybody who was acquainted with him—Mr. Laurence. (Hear, hear.) Mr. Laurence had desired to be relieved from the duties of Treasurer, which he had discharged with so much courtesy and kindness from the foundation of the Institute. (Hear, hear.) The Council had acceded to that desire, and he (Mr. Lodge) could not let the fact of Mr. Laurence's withdrawal pass away without taking this opportunity to ask them to signify amongst themselves, and to communicate to him if thought desirable, the great gratification they felt at having had the advantages of his services for so long a time, and their best wishes for his future welfare. (Hear, hear.) He (Mr. Lodge) had no doubt that in Mr. Cutcliffe they had a gentleman well qualified to undertake the duties of treasurer. (Hear, hear.) Their present business, however, was to thank Mr. Laurence for his past services, and to express their good wishes for his future welfare. He hoped this would meet with the approbation of the meeting. (Cheers.)

Mr. Hodge, in seconding the resolution, said he cordially concurred in what had been said with regard to Mr. Laurence, though he could not agree in the doubt which Mr. Lodge seemed to have as to whether this resolution ought to be communicated to him. He (Mr. Hodge) thought he would be gratified at receiving such a mark of esteem; it was certainly one they ought to pay. (Hear, hear.)

Mr. Brown said he had had many opportunities of communicating with Mr. Laurence during his connection with the Institute, and he should be sorry to allow this opportunity to pass without expressing his gratitude for the many kindnesses he had personally experienced at his hands. The Institute must feel deeply thankful to him for the services he had rendered, and especially for the warm interest he had always taken in their success. (Hear, hear.) He (Mr. Brown) felt much regret at losing his services, but he was glad to say that the cause which led him to retire would not prevent him from taking a warm interest in their proceedings, which he was sure he would always do. (Hear, hear.)

The resolution, which was as follows, was then carried unanimously:—"That the thanks of the members be given to John Laurence, Esq., late Treasurer of the Institute of Actuaries, for the valuable services rendered by him to the Institute during many years."

A vote of thanks was given to the scrutineers.

The CHAIRMAN said they had next to elect three Associates as Auditors. Their present Auditors were Messrs. Berridge, Hardy, and Newbatt—very efficient gentlemen, "all honourable men." (Hear, and laughter.)

Mr. HODGE proposed, and Mr. BROWN seconded, "that the retiring Auditors be re-elected," which was at once carried.

A vote of thanks having been given to the Auditors for their past services,

Mr. NEWBATT felt greatly obliged for the compliment, and especially for the distinguished honour paid to the Auditors by the Chairman in calling them "all honourable men." (Hear, and laughter.) Of course, in these days that was an important consideration, and it would be a very happy thing for a number of people if the same could be said of all auditors. (Hear, hear.)

Mr. CUTCLIFFE then proposed a vote of thanks to the retiring President, Vice-Presidents, Council, and other officers for their services during the past year. He extremely regretted that the President was not present on this occasion, and the more so as his absence was occasioned by indisposition. (Hear, hear.) They were all so well aware of the great interest which Mr. Jellicoe took in this Institute that it was unnecessary for him to refer to it further. (Hear, hear.) The Vice-Presidents, Council, and other officers did everything they could on behalf of the Institute, and were entitled to the warmest thanks of the members. He was extremely glad to find the Institute prospering as it was doing, and to see that the number of members was increasing, which was, after all, the most important consideration. (Hear, hear.)

The resolution was seconded, and carried unanimously.

Mr. BROWN regretted that Mr. Jellicoe was not himself present to return thanks for the kind feeling exhibited towards him and the rest of the officers. He cordially agreed in what had been said with regard to Mr. Jellicoe, than whom no man could take a more hearty interest in their affairs, or be more attentive to those duties which tended to bring about success. (Hear, hear.) He had observed him for many years, and he knew no occasion in which he had not entered warmly into any question upon which the character of the Institute depended. (Cheers.)

A vote of thanks was given to the Chairman for presiding, and the meeting dispersed.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

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*On the Calculation of Life Contingencies.** (*Part II.*) *By* PROFESSOR DE MORGAN.

THE present paper is a continuation of that in the Companion for 1840, on the application of Barrett's method of calculating life contingencies. No tables were then published for contingencies involving two lives; but since that time Mr. Jones's tables have appeared, in his work on the subject in the Library of Useful Knowledge. It may therefore be worth while to collect and describe the formulæ relative to two lives which are most likely to be useful in practice; with some other matters, which may partly interest the actuary and partly the commencing student.

The method described in the former paper is extended by calculating every case of

$$D_{m,n}=a_m a_n v^k,\ D_{m+1,n+1}=a_{m+1}a_{n+1}v^{k+1},\ \&c.$$

where a_x always means the number living at the age x, v^x the present value of £1 due in x years, and k is any convenient number depending on m or n, or both. In the tables published, k is taken to be the greater of the two, m or n; the consequence of which is, that two formulæ are necessary in every case in which the order of survivorship is a part of the question, according as the life, on the survivorship of which a benefit depends, is the elder or younger of the two. This might have been avoided by choosing $\frac{1}{2}(m+n)$ for k, and on such an assumption we shall first exhibit the formulæ which the different cases require; pointing out, at the same time, how to adapt the results to the published tables. Such a course will not only be the most simple with respect to the theory, but will always enable the reader to detect a misprint, if any should exist.

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* See Note, p. 328, vol. xii. This part appeared in the *Companion to the Almanac* for 1842.

We begin, then, as follows:—

$$D_{x,y}=a_x a_y v^{\frac{1}{2}(x+y)},\quad N_{x,y}=D_{x+1,y+1}+D_{x+2,y+2}+\ldots.$$

If the tables were complete we should then have

$$C_{x,y}=(a_x a_y-a_{x+1}a_{y+1})v^{\frac{1}{2}(x+y)+1}$$
$$M_{x,y}=C_{x,y}+C_{x+1,y+1}+\ldots.$$

but as these tables have not been published, we must express M in terms of N as follows:—

$$M_{x,y}=vN_{x-1,y-1}-N_{x,y}=D_{x,y}-(1-v)N_{x-1,y-1}$$

In the former paper $N_{x,y}$ stood for N_x-N_{x+y}; in the present one let

$N_{x,y}-N_{x+s,y+s}$ be denoted* by $N_{x,y|s}$.

In all questions in which the joint duration of two lives takes the place of one life, the formulæ are obtained from those of the former paper by simply substituting $N_{x,y}$ for N_x, and so on. The following are instances, (x) and (y) representing individuals aged x and y years:—

1. The value of an annuity of £1 on the joint lives of (x) and (y) is

$N_{x,y}$ divided by $D_{x,y}$

2. The same, for n years, if both parties live so long, is

$N_{x,y|n}$ or $N_{x,y}-N_{x+n,y+n}$, divided by $D_{x,y}$

3. The value of £1 to be received in k years, if both (x) and (y) be then alive, is

$D_{x+k,y+k}$ divided by $D_{x,y}$

4. The value of an annuity of £1 deferred for k years is

$N_{x+k,y+k}$ divided by $D_{x,y}$

5. The premium for the same (k premiums in all) is

$N_{x+k,y+k}$ divided by $N_{x-1,y-1}-N_{x+k-1,y+k-1}$

Or, to pay for it in l premiums, take $N_{x-1,y-1}-N_{x+l-1,y+l-1}$ for a divisor.

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* It was not contemplated in the last paper that any one would have had the courage to compute tables for two lives. Had it been thought likely, what was there denoted by $N_{x,y}$ would have been denoted by $N_{x|y}$.

6. Present value of an assurance of £1 on the death of the first of (*x*) and (*y*).

$$\frac{M_{x,y}}{D_{x,y}} \text{ or } \frac{vN_{x-1,y-1}-N_{x,y}}{D_{x,y}} \text{ or } \frac{D_{x,y}-(1-v)N_{x-1,y-1}}{D_{x,y}}$$

7. The same, if the first death take place within *n* years.

$$\frac{M_{x,y|n}}{D_{x,y}} \text{ or } \frac{v(N_{x-1,y-1}-N_{x+n-1,y+n-1})-(N_{x,y}-N_{x+n,y+n})}{D_{x,y}}$$

$$\text{or } \frac{D_{x,y}-D_{x+n,y+n}-(1-v)(N_{x-1,y-1}-N_{x+n-1,y+n-1})}{D_{x,y}}$$

8. To pay for either of the preceding in premiums, write for the divisor

$$N_{x-1,y-1} \text{ or } N_{x-1,y-1}-N_{x+l-1,y+l-1}$$

according as the premium is for life or for *l* years.

These are the most common cases, and any others which arise can as easily be found from the preceding paper. The usual cases of reversionary annuities, annuities on the longest liver, &c., may be omitted here, as they are more easily calculated by the common tables.

In cases which involve the order of survivorship, tables in which $\frac{1}{2}(x+y)$ is the exponent of v in $D_{x,y}$, will introduce the square root of v. The method of adapting their results to the case in which that exponent is the *greater of the two*, x or y, must be described as follows:—

Change $D_{x,y}$ into $D_{x,y}v^{-\frac{1}{2}(x-y)}$ or $D_{x,y}v^{-\frac{1}{2}(y-x)}$ according as (*x*) or (*y*) is the older life; and the same for $N_{x,y}$.

In the following instances the first result given is that in which $D_{x,y}$ stands for $a_xa_yv^{\frac{1}{2}(x+y)}$; the second, that in which (*x*) is the older and $D_{x,y}$ denotes $a_xa_yv^x$; the third, that in which (*y*) is the older and $D_{x,y}$ denotes $a_xa_yv^y$.

9. Required the present value and premium of an assurance of £1 on the life of (*x*), provided that (*y*) survive (*x*). The first divisor is for the present value, the second for the premium.

$$\frac{vN_{x-1,y-1}-N_{x,y}+\sqrt{v}(N_{x-1,y}-N_{x,y-1})}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$$(x>y) \quad \frac{v(N_{x-1,y-1}+N_{x-1,y})-(N_{x,y-1}+N_{x,y})}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$$(y>x) \quad \frac{v(N_{x-1,y-1}-N_{x,y-1})+N_{x-1,y}-N_{x,y}}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$(y=x)$ $\frac{1}{2}$ result of (6)

10. If the risk of the preceding question be not to begin until k years have elapsed, write $x+k$ and $y+k$ instead of x and y *in the numerators only,* which then become

$$(x>y)\quad v(N_{x+k-1,\,y+k-1}+N_{x+k-1,\,y+k})-(N_{x+k,\,y+k-1}+N_{x+k,\,y+k})$$

$$(y>x)\quad v(N_{x+k-1,\,y+k-1}-N_{x+k,\,y+k-1})+N_{x+k-1,\,y+k}-N_{x+k,\,y+k}$$

11. And in all cases in which premiums are to be paid l times, instead of during the whole joint duration, the divisor has

$$N_{x-1,\,y-1|l} \text{ or } N_{x-1,\,y-1}-N_{x+l-1,\,y+l-1}, \text{ instead of } N_{x-1,\,y-1}.$$

It will very much simplify both the use and theory of these formulæ if the student make himself perfectly master of the symbol $N_{x,\,y|t}$, which means the value of N at the ages x and y cut off, as it were, at the end of t years, or diminished by all the terms arising from the period beyond t years. Thus

$$\begin{aligned} N_{x,\,y}&=D_{x+1,\,y+1}+\dots\dots+D_{y+t,\,y+t}+D_{x+t+1,\,y+t+1}+\dots\dots\\ N_{x+t,\,y+t}&=\qquad\qquad\qquad\qquad\qquad\quad D_{x+t+1,\,y+t+1}+\dots\dots\\ N_{x,\,y}-N_{x+t,\,y+t}&=D_{x+1,\,y+1}+\dots\dots+D_{x+t,\,y+t}\\ &=N_{x,\,y|t}\end{aligned}$$

12. Required the present value and premium (payable l times) of £1 to be paid at the death of (x), if within n years, and if (y) be then alive.

Cut off every term in the numerator of (9) at n years, and use the divisor for l premiums, when necessary:

$$(x>y)\quad \frac{v(N_{x-1,\,y-1|n}+N_{x-1,\,y|n})-(N_{x,\,y-1|n}+N_{x,\,y|n})}{2D_{x,\,y} \text{ or } 2N_{x-1,\,y-1|l}}$$

$$(y>x)\quad \frac{v(N_{x-1,\,y-1|n}-N_{x,\,y-1|n})+N_{x-1,\,y|n}-N_{x,\,y|n}}{2D_{x,\,y} \text{ or } 2N_{x-1,\,y-1|l}}$$

13. To find the corresponding formulæ in the case in which the assurance is for the death of (x), provided y be then *dead,* calculate the present value of the assurance on the life of (x) absolutely (by the former paper), and subtract the present value of the proper one of the preceding formulæ; the remainder is the present value required. Find the premium by the usual method, or else by multiplying by $D_{x,\,y}$ and dividing by the proper divisor.

14. The present value of a life annuity of n payments on (y), the first being made at the ($k+1$)th anniversary of the present time which follows the death of (x).

N.B. If $k=0$ and n outrun the term of life, this is a common reversionary annuity on (y) after (x).

$$\frac{N_{y+k}-N_{y+k+n}}{D_y}-\frac{N_{x,\,y+k}v^{\frac{1}{2}k}-N_{x,\,y+k+n}v^{\frac{1}{2}(k+n)}}{D_{x,\,y}}$$

To suit the published tables the second numerator is to be altered as follows, all the rest remaining:—

$x <$ or $= y$ into $N_{x,y+k} - N_{x,y+k+n}$
$x > y <$ or $= y + k$ ” $(N_{x,y+k} - N_{x,y+k+n})v^{x-y}$
$x > y + k <$ or $= y + k + n$ ” $N_{x,y+k}v^{k} - N_{x,y+k+n}v^{x-y}$
$x > y + k + n$ ” $N_{x,y+k}v^{k} - N_{x,y+k+n}v^{k+n}$

15. The present value of £1 to be paid at the death of (x), if he die before (y) or within n years after (y).

$$\frac{M_x - M_{x+n}}{D_x} +$$

$$v^{\frac{1}{2}n}\frac{N_{x+n-1,y-1}v - N_{x+n,y} + v^{\frac{1}{2}}(N_{x+n-1,y} - N_{x+n,y-1})}{2D_{x,y}}$$

To suit the published tables the second numerator ($v^{\frac{1}{2}n}$ included) must be

$$(x+n<y)$$

$$(N_{x+n-1,y-1} - N_{x+n,y-1})v^{n+1} + (N_{x+n-1,y} - N_{x+n,y})v^{n}$$

$$(x<y,\ x+n>y)$$

$$\{(N_{x+n-1,y-1} + N_{x+n-1,y})v - (N_{x+n,y} + N_{x+n,y-1})\}v^{y-x}$$

$x > y$ $\quad (N_{x+n-1,y-1} + N_{x+n-1,y})v - (N_{x+n,y} + N_{x+n,y-1})$

If this risk be to continue only for $n+p$ years, all the Ns must be stopped after p years, or $N_{x+n-1,y-1\,|\,p}$ must be written for $N_{x+n-1,y-1}$, and so on.

If $x+n=y$, the second numerator must be

$$\{(N_{x+n-1,y-1}v + N_{x+n-1,y}) - (N_{x+n,y} + N_{x+n,y-1})\}v^{n}$$

16. The present value of £1 to be paid on the death of (x), provided he die within n years after y.

From the preceding result subtract the present value of £1 payable at the death of (x), if (y) survive. (See No. 9.)

17. The present value of £(1) to be paid at the death of (x), if he survive (y).

From the present value of an assurance on the life of (x) subtract the result of (9), or the value of £1 to be paid at his death, if y survive.

18. The present value of £1 to be paid at the death of (x), if more than n years after the death of (y).

From the present value of an assurance of £1 on the life of (x) subtract the result of (15). This gives $M_{x+n} \div D_x$ for the first term.

I have not thought it necessary to write down the formulæ connected with the last few cases, as they seldom occur.

There are, we are told, still some persons who cannot see how much shorter this method of Barrett's is than that in previous use. It is certain that any new method does not become so easy as the old and well-known one until a few examples have been tried; and there are dispositions which will not give a new method a fair trial. But, for those who may be commencing the subject, and to whom, therefore, all methods stand entirely on their own merits, we subjoin the example of a simple temporary assurance on two joint lives, worked by both methods.

Required the present value of an assurance for the next ten years only, on the joint continuance of two lives, now aged 45, and 50 years: interest 5 per cent. (Milne, p. 341.)

Old Method.

a_{55}	3·60991	1—·43827	
a_{60}	3·56146	=·56173	$\bar{1}$·74953
a^{-1}_{45}	$\bar{4}$·32541	v	$\bar{1}$·97881
a^{-1}_{50}	$\bar{4}$·35684		
v^{10}	$\bar{1}$·78811	·53498	$\bar{1}$·72834
·43827	$\bar{1}$·64173	6·623	·82105
	·85163	$1-v$	$\bar{2}$·67778
3·114	·49336	·31538	$\bar{1}$·49883
9·737		·53498	
6·623		·21960 present value of £1.	

Barrett's Method.

$N_{44,49}$	19460646		$N_{45,50}$	17648150	
$N_{54,59}$	6438764		$N_{55,60}$	5644407	
	13021882	7·11468		12003743	
	$1-v$	$\bar{2}$·67778			
	620100	5·79246			
	12401782				
	12003743				
	398039	5·59992			5·59992
$D_{45,50}$	1812496	6·25828		13021882	7·11467
Present value	·21960	$\bar{1}$·34164	Premium	·030567	$\bar{2}$·48525
			Milne, p. 346,	·030565	

If we were to look at the number of figures only there would seem to be little difference between the two methods, the first containing about 130 figures and the second 120. But if we look at the number of entries of different tables in the two, we find it as follows:—

Old Method.		*New Method.*	
No. of pages in Milne which must be consulted for log a_{55}, &c., v^{10}, 9·737, v and $1-v$	3	No. of pages in Jones to be consulted for $N_{44,59}$.... ..$D_{45,50}$	1
Logarithms to be taken out .	6	Logarithms taken out . . .	4
Numbers to logarithms . .	4	Numbers to logarithms . . .	2
	13		7

In multiplying by v in the above process, the formula taken is

$$av=a-(1-v)a,$$

and $(1-v)a$ is found by logarithms and subtracted from a. The reason is that v is so near unity that it would take tables of logarithms to seven places to get five places of the answer correct, if av were to be directly found.

As another example, take the following (Milne, p. 360): Required (at 5 per cent.) the annual premium for £1, payable at the death of (50), if within 10 years, and (45) be then alive; premiums being payable for ten years, if the joint lives endure so long. The first formula of No. 12 must be employed:

$N_{49,44}$ 19460646	$N_{50,44}$ 18016514	$N_{49,45}$ 19057899	$N_{50,45}$ 17648150
$N_{59,54}$ 6438763	$N_{60,54}$ 5834103	$N_{59,55}$ 6227029	$N_{60,55}$ 5644407
13021883	12182411	12830870	12003743
12830870			12182411
25852753	7·4125067		24186154
$1-v$	$\bar{2}$·6777803		
1231082	6·0902870		
24621671			
24186154			
435517	5·6390051		
13021883	7·1146737		
2)·03344502	$\bar{2}$·5243314		

·01672251 required premium.

Let the problem be now changed by simply inverting the ages. Required the annual premium for £1 payable at the death of (45) if within ten years, and (50) be then alive. The second formula of No. 12 must now be used, but our preceding process contains all the data:

$N_{44,49\mid 10}$	13021883	$N_{44,50\mid 10}$	12182411
$N_{45,49\mid 10}$	12830870	$N_{45,50\mid 10}$	12003743
	191013	5·2810629	178668
	v	$\bar{1}$·9788107	
	181917	5·2598736	
	178668		
	360585	5·5570076	
	13021883	7·1146737	
	2)·02769069	$\bar{2}$·4423339	
	·01384534	present value required.	
	·01672251	result of the preceding.	
	·03056785		

This last result ·03056785 is the premium for the assurance on the joint lives, and agrees both with the above result and with Mr. Milne. But Mr. Milne, instead of ·01672251, has (p. 361) ·015714, differing materially from the truth. The reason is, as those who are used to the common method are well aware, that the old tables of annuities, calculated to three decimal places, are not sufficiently exact for survivorship questions, the results of which depend upon the differences of nearly equal annuities, particularly where, owing to the differences of ages given being only every five years, the defective interpolation by first differences is employed. Mr. Milne's process, it is hardly necessary to say, is as good as, with his tables, it could have been: but the above examples show, not only that Barrett's method is much the more easy of the two, but also the more exact, the tables actually published* being used. All the above results are true to six significant figures.

We now proceed to consider one or two questions in which approximations are usually made, with a view to state the results of an examination of the approximations, and to propose some amendments upon them.

1. The values of life annuities are usually calculated on the supposition that interest can be converted into principal only once a year, the truth being that such conversion always takes

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* The superior exactness of the above method, as it stands, is not owing to anything in the method, but solely arises from the greater extent of the tables published.

place twice a year at least. Mr. Milne has given a rule (p. 367) for making the requisite correction, but I believe the following will be found to be more easy of application: it is given for the case of half-yearly conversions only. Let the value of an annuity (for lives or a term of years) be A at c per cent., A_1 at $c+1$ per cent., A_2 at $c+2$ per cent., and so on. Take the differences of A, A_1, &c., as follows:—

$$\begin{array}{lllllll} A & & & & & & \\ & B & & & B=A-A_1 & B_1=A_1-A_2 & B_2=A_2-A_3 \\ A_1 & & C & & & & \\ & B_1 & & D & C=B-B_1 & C_1=B_1-B_2 & \\ A_2 & & C_1 & & & & \\ & B_2 & & & D=C-C_1 & & \\ A_3 & & & & & & \end{array}$$

Then the value of the same annuity at c per cent., interest being convertible twice a year, is

$$A-\frac{c^2}{400}\left(B+\frac{C}{2}+\frac{D}{3}-\frac{c^2(C+D)}{800}\right), \text{ very nearly.}$$

Example 1. Required the value of a perpetual annuity of £1 at 4 per cent., payable yearly, on the supposition that money is convertible half yearly.

$$\begin{array}{llll} A\ =25{\cdot}000 & & & \\ & 5{\cdot}000 & & \\ A_1=20{\cdot}000 & & 1{\cdot}667 & \\ & 3{\cdot}333 & & {\cdot}715 \\ A_2=16{\cdot}667 & & {\cdot}952 & \\ & 2{\cdot}381 & & \\ A_3=14{\cdot}286 & & & \end{array}$$

$$\begin{array}{lr} (1{\cdot}667+{\cdot}715)\frac{16}{800}={\cdot}048 & 5{\cdot}000 \\ & {\cdot}889 \\ 6{\cdot}079\times\frac{16}{400}={\cdot}243 & {\cdot}238 \\ & \overline{6{\cdot}127} \\ & {\cdot}048 \\ & \overline{6{\cdot}079} \end{array}$$

$$25{\cdot}000-{\cdot}243=24{\cdot}757 \text{ answer.}$$

The real answer is 24·75225, and a perpetual annuity is the severest trial which can be made of the rule.

Example 2. (Milne, p. 367.) Required the value of a life annuity on (60), at 5 per cent., interest being convertible twice a year:

$$\begin{array}{lllll} A & 8{\cdot}940 & & & \\ & & {\cdot}636 & & \\ A_1 & 8{\cdot}304 & & {\cdot}075 & \\ & & {\cdot}561 & & {\cdot}012 \\ A_2 & 7{\cdot}743 & & {\cdot}063 & \\ & & {\cdot}498 & & \\ A_3 & 7{\cdot}245 & & & \end{array}$$

$$\begin{array}{lr} ({\cdot}075+{\cdot}012)\frac{25}{800}={\cdot}0027 & {\cdot}6360 \\ & {\cdot}0375 \\ {\cdot}6748\times\frac{25}{400}={\cdot}04218 & {\cdot}0040 \\ & \overline{{\cdot}6775} \\ & {\cdot}0027 \\ 8{\cdot}940-{\cdot}04218=8{\cdot}89782 & \overline{{\cdot}6748} \end{array}$$

Mr. Milne's answer is 8·8977.

2. When an annuity is to be paid, not yearly, but half-yearly or quarterly, &c., interest being convertible into principal only once a year, the common correction of A, the present value of a yearly annuity, is to write $\mathrm{A}+\frac{n-1}{2n}$ instead of A, n being the number of times which the annuity is payable yearly. This correction is strictly true when we speak of a perpetual annuity, on the supposition that by interest being convertible once a year, we mean that money may be invested at any one time of the year, so that the grantor of the annuity must be held to lose simple interest for the rest of the year on each of the prepaid portions. When this correction is to be investigated for a life annuity, it would generally be supposed that those who die in each year die at equal intervals. This is an insufficient supposition; for if the mortality of the table be increasing from year to year, it ought to be supposed that the mortality of the latter part of a year is greater than that of the former. We have therefore taken the supposition that the mortality of each year is to be constructed on the hypothesis of constant third differences, which has the effect of distributing the deaths of each year in a manner depending on the mortality of the preceding and following years.

Proceeding on this plan we find the following result. Let the annuity be payable n times a year, let the age be k, and let a_k be the number in the table alive at that age, A_k being the tabular yearly annuity. Also let

$$\beta=\frac{n-1}{2n},\qquad \beta'=\frac{n^2-1}{72n^2},\qquad \beta''=\frac{n^4-1}{180n^4}.$$

The value* of the annuity then is, accurately, so far as the suppositions are accurate (r being the interest of £1 for one year)

$$\left(1+6\beta'\frac{r^2}{1+r}-(\beta'+\beta'')\frac{r^4}{(1+r)^2}\right)\mathrm{A}_k$$

$$+\beta-(8\beta'+2\beta'')\frac{r}{1+r}-(\beta'+\beta'')\frac{r^2}{(1+r)^2}$$

$$-3\beta'\frac{a_{k-1}-a_{k+1}}{a_k}+(\beta'+\beta'')r\left(\frac{a_{k+1}}{a_k}+\frac{1}{1+r}\frac{a_{k-1}}{a_k}\right).$$

This expression contains a coefficient for A_k depending (n apart) on the rate of interest only; certain terms which depend on the rate of interest only; another depending on the mortality only; and others depending jointly on the rate of interest and the mortality. To form an idea of the utmost value of these corrections, let us take the extreme case of an annuity payable momently (or n infinite) and a high rate of interest, 10 per cent. We have then

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* As we give no demonstration, we should say that, besides obtaining this expression from demonstration, we have tried several verifications: the simplest is that in which a_k is a constant, and $\mathrm{A}_k=1:r$, which gives the perpetual annuity.

$$\beta=\frac{1}{2}, \qquad \beta'=\frac{1}{72}, \qquad \beta''=\frac{1}{180};$$

and the value of the annuity is

$$1{\cdot}00076\,A_k+{\cdot}48873-{\cdot}03989\frac{a_{k-1}}{a_k}+{\cdot}04361\frac{a_{k+1}}{a_k},$$

$A_k+{\cdot}5$ being the common approximation.

We may say then that the common approximation is generally correct to the first place of decimals, but by no means to the second or third. Now, it may fairly be insisted, looking at the manner in which such calculations are usually conducted, that the second place at least in the decimal part of an annuity shall be made as correct as can be conveniently done; not that this is, generally speaking, of very great consequence, but because there are other things by the dozen of no greater consequence, which are always attended to.

3. If in the preceding money make no interest, A_k becomes $(a_{k+1}+a_{k+2}+\ldots)$ divided by a_k, and the annuity represents the mean duration of life, which is therefore (n being infinite)

$$\frac{a_{k+1}+a_{k+2}+\ldots}{a_k}+\frac{1}{2}-\frac{1}{24}\frac{a_{k-1}-a_{k+1}}{a_k},$$

the first two terms are those from which the mean duration is usually deduced: the third is substantially the correction which we have elsewhere* proposed.

4. An annuity for a term of years must be computed in the usual way from the corrected life annuities.

5. No tables have been published for the mean duration of pairs of lives. This defect may be approximately supplied (see the work last cited, Appendix, p. xxx.), but much better from Mr. Jones's Tables of Annuities at 3, 3½, 4, 4½, and 5 per cent. If these be called A_6, A_7, A_8, A_9, and A_{10}, the approximate formula is

$$105(2A_6+9A_8)+126A_{10}-80(9A_7+7A_9)+{\cdot}5.$$

Example 1. Required the mean duration of a life aged 45 (Carlisle Table).

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* Cab. Cyclop. Probabilities and Life Contingencies, p. 163.

$(2A_6+9A_8=158{\cdot}671)\times 105=$	16660·455	15942·885
$(A_{10}=12{\cdot}648)\times 126=$	1593·648	1521·828
	18254·103	17464·713
$(9A_7+7A_9=227{\cdot}870)\times 80=$	18229·600	17440·160
	24·503	24·553
	·5	·5
	25·003	25·053
Correct answer	24·46	

Example 2. Supposing male and female life not to be of different value, what is the average duration of a marriage between parties aged 30 and 25?

Answer, 25 years, calculated as above on the right.

Example 3. What is the average age at which a son of 7 years succeeds to the estate of a father of 37, and how long on the average does he enjoy it?

The first part of the question means the average age of those sons who do succeed, without any reference to those who do not survive their fathers; and since at 37 the mean duration of life is 30 years, the answer is that such sons succeed, one age with another, at the age of 37. The last part of the question cannot be directly solved by existing tables, if it refer to the actual average period of enjoyment of all those persons who live to come into possession; but it must first be found how many years of enjoyment those sons have, one with another, on the supposition that the actual years of enjoyment are divided among the whole, both those who live to come into possession and those who do not. If L_x be the mean duration of (x), and $L_{x,y}$ that of (x) and (y) jointly, then $L_7-L_{7,37}$ is *approximately* the mean period by which a son of 7 years survives a father of 37. Or, if L_x and $L_{x,y}$ represent the values of the annuities when money makes no interest, $L_x-L_{x,y}+{\cdot}5$ is the mean period of survivorship; then $L_{x,y}$ is calculated by the above method, omitting the ·5, and $L_x+{\cdot}5$ is the mean duration of (x) in the table. Calculate $L_{7,37}$, which gives 27·62, and L_7 being 50·30, we have 22·68 (say 23) years for the average term of enjoyment. That is to say, if 1000 sons, each aged 7, and the father of each being aged 37, were to agree to club their future years of possession in such manner that all (or their executors) should take equal shares, whether they lived to come into possession or not, the years of enjoyment of those who do come into possession would yield to each of the 1000 sons

23 years a-piece, or thereabouts; and of course the average term of those who *do* come into possession, nothing being supposed given up, is considerably larger than this; and an approximation (and only an approximation, though very plausible-looking reasoning might be put together to make it appear an exact method) may be made to it as follows: Divide this term of 23 years by the chance which a son of 7 years has of surviving a father aged 37. In the table of chances of survivorship, it appears that ·7837 is the chance of (7) surviving (37), and $23 \div \cdot 7837 = 29$, which is about the number of years required. This approximation always gives the result too great, perhaps by a year or two; but rough as the method may be, it is, as far as we know, the first which has been proposed—the want of that most essential preliminary, tables of mean survivorship of two lives, having prevented the subject from being considered.

Example 4. What would be the joint duration of twins if they were assumed to be, at birth, lives of the same value as other infants? The calculation gives 21 years.

Example 5. What is the mean duration of the survivor of (25) and (30)? This must be found, approximately, from the formula $L_{25} + L_{30} - L_{25, 30} + \cdot 5$, or $37{\cdot}4 + 33{\cdot}8 - 24{\cdot}6 + \cdot 5$, or 47 years.

Example 6. What is the mean joint duration of the three lives (7), (24), and (40)? Take the two elder lives and calculate their mean joint duration, which is found to be 26·5. Find a life which has this mean duration, which is (42), and find the mean joint duration of (7) and (42) or 27 years. This result is absurd, since the mean duration of (42) itself is not quite 27 years; it arises from the error of the process. In the present instance the mean duration of (7) is upwards of 50 years, so that the joint duration of (7) and (42) is very little less than that of (42) itself; so little less, that the error incidental to the process (which is generally in excess) more than makes the difference.

Also the annuities used in the application of this rule must be tolerably exact to three decimals; and if there be any doubt about the interpolated annuities, recourse must be had to those derived from Barrett's tables.

6. Supposing an individual (x) to lay by £1 every year, what may he reasonably count upon having at his death, supposing that he improves the money at compound interest?

Supposing £1 to be laid by now, the question is the same as, how much should a premium of £1 assure at his death? and the answer is ;—N_{x-1} divided by M_x in the table, or $(1+r)(1+A)$

divided by $1-rA$ from the common table, A being the value of a life annuity of £1. We mention this here because there is a table in Mr. Morgan's "Treatise on Assurances" (Table III., described in p. 54, and note V.) which was copied in part into the "Penny Cyclopædia," as being what its name would import, "the sum to which an annuity forborne and improved will amount on the extinction of a given life." Mr. Morgan's table gives the answer to the following question; if every one at his death were to receive his annuity (not a premium; exclude therefore the immediate payment) improved to its amount at the beginning of the year, what would persons of the same age at commencement receive one with another, from all the different amounts which they actually do receive? In our notation it is a calculation of the value of

$$\frac{a_{x+1}-a_{x+2}}{a_x}+\frac{a_{x+2}-a_{x+3}}{a_x}(1+\overline{1+r})+\frac{a_{x+3}-a_{x+4}}{a_x}(1+\overline{1+r}+\overline{1+r}^2)+\ldots$$

or,
$$\frac{a_{x+1}}{a_x}+\frac{a_{x+2}}{a_x}(1+r)+\frac{a_{x+3}}{a_x}(1+r)^2+\ldots.$$

This table may therefore serve (at least by the Northampton table at 4 per cent.) to compare the following cases. A and B are both of the same age: A is to assure his life immediately at a premium of £1, and B will in a year lay by £1, and the same at the end of every subsequent year. Supposing each to get the average of his class at death, what will they severally get? To bring both accounts up to the end of the year of death, at which the common assurance is supposed to be paid, a year's interest must be added to the result in Mr. Morgan's table. The following are the average sums obtained in the two cases, at different ages:—

	A	B
20	49·4	98·3
25	44·7	82·2
30	40·2	68·6
35	35·7	56·8
40	31·3	46·7
45	27·2	38·0
50	23·2	30·7
55	19·7	24·5
60	17·0	19·2

Supposing the Northampton table and 4 per cent. to be facts (we have chosen this table and rate because the only deduced table of the kind we are considering is Mr. Morgan's), it thus appears that the office gets from persons aged 20, one with another, in premiums and interest, £98·3 and the first premium with its interest, from each person, for every £49·4 assured. But many of these persons receive each his £49·4 long before he has earned it by his premiums, and the advances thus made by the office must be repaid with interest by those who live longest.

7. The effect of compound interest, as distinguished from simple, is not very easily made apparent, and is not of great importance, since simple interest, as applied to annuities, is really a fiction, not more worthy of attention than would be the transactions of a man who, having a number of sovereigns in his hand, and others coming in, should invent a rule for separating some from the rest, and locking them up in a cash-box, instead of investing them. Some writers have argued that the *theory* of annuities at simple interest must be wrong, inasmuch as its consequence is, that an infinite sum is necessary to pay a perpetual annuity; the simple truth being, that if any one were to carry that theory into practice, he would find that the prediction of theory was verified; namely, that no sum, however great, would enable him to pay a perpetual annuity. According to the theory, the present value of an annuity (the rate of interest being r per pound) of £1 for n years is

$$\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}+\dots\dots+\frac{1}{1+nr}.$$

If n be considerable, the labour of summation will be materially lessened by using the following approximation.

Let $\frac{1}{1+r}=P_1$, and $\frac{1}{1+nr}=P_n$

the value of the preceding series is nearly (the logarithm used being the common one)

$$\frac{2{\cdot}3025851}{r}\log\frac{1+nr}{1+r}+\frac{1}{2}(P_1+P_n)+\frac{r}{12}(P_1^2-P_n^2)-\frac{r^3}{120}(P_1^4-P_n^4).$$

To try this approximation, suppose r as great as $\cdot 1$ and $n=10$. We have then $1:r=10$, $nr=1$ (the powers of P_1 may be taken from the table of present values of £1).

$$\left(\log\frac{2}{1{\cdot}1}={\cdot}2596373\right)\times\frac{2{\cdot}3025851}{{\cdot}1}=\quad 5{\cdot}978370$$

$$\frac{1}{2}({\cdot}909091+{\cdot}5)\quad=\quad{\cdot}704546$$

$$\frac{{\cdot}1}{12}({\cdot}826446-{\cdot}25)\quad=\quad{\cdot}004804$$

$$=\frac{({\cdot}1)^3}{120}({\cdot}683013-{\cdot}0625)\quad=-\quad{\cdot}000005$$

$$6{\cdot}687715$$

		Reciprocal.
Verification.	1·1	·90909091
	1·2	·83333333
	1·3	·76923077
	1·4	·71428571
	1·5	·66666667
	1·6	·62500000
	1·7	·58823529
	1·8	·55555556
	1·9	·52631579
	2·0	·50000000
		6·68771403
	Approx.	6·687715
	Error	0·000001

The approximative process would not be more difficult, nor less exact, if the number of terms were a million.

8. A more practical question than the last is derived from the following considerations. It is well known that individuals rarely or never make compound interest of what they lay by; while it is possible that many lay by a sum which increases from year to year equally or nearly so. We have seen (in 6) that an assurance office seems to offer a greater moral benefit to the old than the young; inasmuch as the latter, in making the office their trustee, seem to give up a more brilliant average prospect. But this is on the supposition that compound interest is made: let us now suppose that the *office* makes compound interest, while the *individual* would only lay by an increasing annual sum. We now introduce the following table. (See p. 145.)

This table is nothing but the table described in our last paper, on the supposition that money makes no interest, so that all results depend merely on the average duration of life. This table, as we shall proceed to show, concerns persons in general more almost than any other of the kind. All persons speculate, more or less, on what they can do with their incomes during their lives, without thinking of interest; that is, they attempt to form notions which this table would enable them to form justly.

Example 1. A person aged x is to lay by £1 at the end of every year from this time. What will the savings amount to at his death, one person with another?

Ans. $\frac{N_x}{D_x}$; if $x=20$, $\frac{249432}{6090}=£40\cdot96$.

Age.	D.	N.	S.	Age.	D.	N.	S.
0	10000	382213	12603644	53	4211	77784	967464
1	8461	373752	12221431	54	4143	73641	889680
2	7779	365973	11847679	55	4073	69568	816039
3	7274	358699	11481706	56	4000	65568	746471
4	6998	351701	11123007	57	3924	61644	680903
5	6797	344904	10771306	58	3842	57802	619259
6	6676	338228	10426402	59	3749	54053	561457
7	6594	331634	10088174	60	3643	50410	507404
8	6536	325098	9756540	61	3521	46889	456994
9	6493	318605	9431442	62	3395	43494	410105
10	6460	312145	9112837	63	3268	40226	366611
11	6431	305714	8800692	64	3143	37083	326385
12	6400	299314	8494978	65	3018	34065	289302
13	6368	292946	8195664	66	2894	31171	255237
14	6335	286611	7902718	67	2771	28400	224066
15	6300	280311	7616107	68	2648	25752	195666
16	6261	274050	7335796	69	2525	23227	169914
17	6219	267831	7061746	70	2401	20826	146687
18	6176	261655	6793915	71	2277	18549	125861
19	6133	255522	6532260	72	2143	16406	107312
20	6090	249432	6276738	73	1997	14409	90906
21	6047	243385	6027306	74	1841	12568	76497
22	6005	237380	5783921	75	1675	10893	63929
23	5963	231417	5546541	76	1515	9378	53036
24	5921	225496	5315124	77	1359	8019	43658
25	5879	219617	5089628	78	1213	6806	35639
26	5836	213781	4870011	79	1081	5725	28833
27	5793	207988	4656230	80	953	4772	23108
28	5748	202240	4448242	81	837	3935	18336
29	5698	196542	4246002	82	725	3210	14401
30	5642	190900	4049460	83	623	2587	11191
31	5585	185315	3858560	84	529	2058	8604
32	5528	179787	3673245	85	445	1613	6546
33	5472	174315	3493458	86	367	1246	4933
34	5417	168898	3319143	87	296	950	3687
35	5362	163536	3150245	88	232	718	2737
36	5307	158229	2986709	89	181	537	2019
37	5251	152978	2828480	90	142	395	1482
38	5194	147784	2675502	91	105	290	1087
39	5136	142648	2527718	92	75	215	797
40	5075	137573	2385070	93	54	161	582
41	5009	132564	2247497	94	40	121	421
42	4940	127624	2114933	95	30	91	300
43	4869	122755	1987309	96	23	68	209
44	4798	117957	1864554	97	18	50	141
45	4727	113230	1746597	98	14	36	91
46	4657	108573	1633367	99	11	25	55
47	4588	103985	1524794	100	9	16	30
48	4521	99464	1420809	101	7	9	14
49	4458	95006	1321345	102	5	4	5
50	4397	90609	1226339	103	3	1	1
51	4338	86271	1135730	104	1	0	0
52	4276	81995	1049459				

Example 2. A person aged x is to lay by £1 for n years, if he live so long: what will be the average amount?

$$\textit{Ans.}\quad \frac{N_x - N_{x+n}}{D_x};\ \text{if } x=20,\ y=10,\ \frac{249432-190900}{6090}=£9{\cdot}6.$$

Example 3. What should a person aged x lay by now and at the end of every year, that, one such person with another, there may be £1 at death?

$$\textit{Ans.}\quad \frac{D_x}{N_{x-1}};\quad x=20,\quad \frac{6090}{255522}=£{\cdot}024,\ \text{say } 0s.\ 6d.$$

Example 4. A person aged x is to lay by a at the end of a year, $a+h$, $a+2h$, &c., at the end of 2, 3, &c., years; what will be the average savings of such persons, one with another?

$$\textit{Ans.}\quad a\frac{N_x}{D_x}+h\frac{S_{x+1}}{D_x}.$$

This problem represents better than any other in our opinion, the *best* supposition that can be made as to what any individual can do who attempts to be his own life-assurer. If $x=20$, we have $41a+990h$ for the average required.

Suppose then an individual (A) in this way steadily to invest £1 every year, with three per cent. simple interest upon the preceding investments, so that $a=1$, $h={\cdot}03$, and the preceding is £71; such individuals, one with another, will have £71 at death; that is, some will have next to nothing, a majority less (many of them much less) than £71, and a minority more, a few of them much more. Another individual (B) goes with £1 down and £1 yearly to an office which charges the Carlisle premiums at 3 per cent., increased by 25 per cent. for management and fluctuation. Such an office, returning profits, will pay, let us say £120 for every £100 assured, to its customers, one with another. If this be the case it will pay the preceding individual £64 at his death. Let us now see how the parties stand.

A	B
has been gambling, setting a remote chance of more than the office would give against a great risk of leaving next to nothing, or at any rate very much less than the policy. He has had all the trouble of investing his	has exchanged chance for certainty at a very moderate sacrifice, owing to the office being able to gain compound interest. He has no trouble in investing, beyond that of a walk once a year to the office, and is very

savings, all the risk of being led to engage in some foolish speculation* which should lose them, and if neither his industry nor his prudence should fall short, all the anxiety of leaving too little behind him for many years. *If* he succeed, he may succeed very signally, and he may have the compound interest of his savings, if he know how to make it; but it is much more than an even chance that he does not do nearly so well as B.

nearly rid of temptation to change his investment by knowing that he can only do it at a loss, and by having felt the comfort of knowing himself secure. He has, from the very beginning, a certainty of having made sure of his average, a state of mind highly conducive to health. His success is moderate, but certain; and it never surprises him if, any day of any week, he is solicited to aid in saving from starvation the widow and children of A.

9. We shall end this paper with the consideration of a question which is of frequent occurrence, namely, the borrowing of money to be paid by annuity. A person aged x (the annuity on his life being worth A_x year's purchase) borrows £1, for which he is to pay a life annuity. This life annuity ought to be $1 : A_x$ of a pound. Now if p_x be the premium of assurance for £1 at the required age, we have, r being the interest of £1 for one year,

$$\frac{1}{1+A_x}-\frac{r}{1+r}=p_x, \qquad \text{or } \frac{1}{A_x}=\frac{r+(1+r)p_x}{1-(1+r)p_x}.$$

First, suppose an assurance office to exist, charging the absolute premium of the tables, and nothing more. Let the lender insure the borrower's life, and take from him a yearly payment equivalent both to the interest of the sum lent and the premium. Remember that the lender pays the premium at the beginning of the year, while the borrower's annuity payment does not become due till the end, whence the borrower must pay interest for each premium advanced. This he does (with the interest of the loan) as long as he lives, and, at the end of the year of death, the lender claims from the office: 1, the £1 lent; 2, a year's interest on it, just due; 3, the premium last paid; 4, a year's interest due on it. Let S be the sum assured; then S must be made up of 1, r, p_xS (its premium), and rp_xS a year's interest. Hence

$$S=1+r+p_x(1+r)S, \quad \text{or } S=\frac{1+r}{1-p_x(1+r)}=\frac{1}{v-p_x}.$$

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* In particular, the temptation of spending the *interest* of his savings in making a greater appearance, a folly which has prevented many a competency from being acquired.

which is the sum the lender must assure, to keep himself clear of all loss. Now the borrower must pay at the end of each year, 1, the interest just due on £1; 2, the preceding premium advanced for him a year before, with interest, so that his annuity must be

$$r+\frac{(1+r)^2}{1-p_x(1+r)}\cdot p_x, \quad \text{or} \ \frac{r+(1+r)p_x}{1-(1+r)p_x}, \quad \text{or.} \ \frac{1}{v-p_x}-1,$$

as found before. And this last process* is true, whether the office charges the tabular premium, or makes an addition.

Let us now examine how far the existence of assurance offices affects the position of a borrower. Formerly, in borrowing on annuity, the usury laws made the position of the lender hardly respectable, for his risk (as no one individual could have many transactions of the kind, so as to secure an average) required what might either be called a high rate of mortality or a high rate of interest, at the pleasure of the lender; for the price which supposed 4 per cent., deduced from one table, might be the price at 8 per cent. deduced from another. It is certain that, with the money-lenders, 10 per cent. on the Northampton table was not uncommon. The life of a man of 35 can be respectably insured for £2. 10*s*. per £100, or £·025 per £1. If he borrow at 5 per cent. in the preceding manner, we have then $r=\cdot05$, $p_x=\cdot025$, and £·0783 is the value of the preceding expression, say, £78. 6*s*. for £1,000. Now, if this man had gone to a money-lender who asked 10 per cent., even taking the Carlisle table (which is favourable in this case to the borrower), he must have paid an annuity of £116. 11*s*. for the same accommodation. And in many other ways it might be shown that the existence of assurance-offices is a most valuable protection, not as enabling persons to borrow (an objection we have before now heard urged against them), for borrowing was always practicable, but as enabling them to borrow on reasonable terms, and of respectable people; which the usury laws had rendered almost impracticable.

This transaction must be arranged as follows, when it is desired to pay off the loan in n years. Let p be the premium for assuring £1 for n years, k that for endowing the debtor with £1, if he live over n years. The lender must then (the loan being supposed £1) insure the debtor's life for $\frac{1+k}{v-p}$, and must also pay a premium for £1, to be paid by the office if the debtor survive n years. This premium to the office will therefore be

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* As far as I know, this exact method is due to Mr. Griffith Davies, and the first proof of it is in Mr. Jones's work (p. 189).

$$\frac{1+k}{v-p}p+k, \text{ or } \frac{p+vk}{v-p}.$$

And the debtor's annuity is $\frac{1+k}{v-p}-1$, the first payment being due in a year, and the last at the end of the n years, if he live so long.

A. DE MORGAN.

University College, June 21, 1841.

⁂ Mr. Peter Gray has favoured us with the following errata in our reprint of the first Number:—

Page 334,	line 25,	for	$+m-1)\beta$,	read	$+(m-1)\beta$.
,, 335,	,, 6,	,,	$\theta R_{x+1,}-\nu\varpi R_{x,\ +1}$,	,,	$\theta R_{x+1,n}-\nu\varpi R_{x,n+1}$.
,, 338,	,, last,	,,	comes,	,,	becomes.
,, 340,	,, 18,	,,	$a(hn=a)$,	,,	$a, (hn=a)$.
,, ,,	,, 20,	,,	*dccreasing*,	,,	*decreasing*.
,, 341,	,, 35,	,,	and,	,,	into.
,, 344,	,, 14,	,,	$\mu-1$,	,,	$\mu=1$.
,, 345,	,, 26,	,,	portion,	,,	proportion.
,, 348,	,, 6,	,,	Y,	,,	$Y\varpi$.

On the Construction of Tables by the Method of Differences. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

SECTION II.—*On the Calculus of Finite Differences.*

(50). The Calculus of Differences is a somewhat extensive subject, but it is not necessary now to go far into it, or to occupy ourselves with its application to other than the simplest class of functions, namely, rational, algebraical, and integer functions. The theorems we require might have been left to be deduced as occasion for them should arise; but it has been thought better—and the arrangement will certainly be more convenient for reference—to bring the greater portion of them together in this place.

(51). In this calculus functions of a variable, as x, are denoted by the letters u, v, &c., with the variable attached. Thus, u_x, v_x, &c., denote functions of x, and u_{x+n}, v_{x+n}, &c., denote the same functions, respectively, when x in them is changed into $x+n$. Also, when x takes a particular value, 10 for example, they will be denoted by u_{10}, v_{10}, &c.

(52). If in any function, u_x, x is changed into $x+1$, the excess, positive or negative, of u_{x+1} over u_x is called the difference of u_x, and is denoted by Δu_x. We thus have

$$\Delta u_x = u_{x+1} - u_x.$$

(53). From this it appears that Δ is the symbol of the operation that consists in increasing x in u_x by unit, and taking the difference between the function as changed, and the original function; and Δu_x denotes the result of this operation.

(54). The following are examples of differences:—

If $u_x = x \therefore \Delta u_x = (x+1) - x = 1.$

„ $u_x = 3x \pm 6 \therefore u_x = \{3(x+1) \pm 6\} - (3x \pm 6)$
$= 3(x+1) - 3x = 3.$

„ $u_x = 3x^2 + 4x - 6,$
$\therefore \Delta u_x = \{3(x+1)^2 + 4(x+1) - 6\} - (3x^2 + 4x - 6)$
$= 3(x+1)^2 - 3x^2 + 4(x+1) - 4x$
$= 3x^2 + 6x + 3 - 3x^2 + 4x + 4 - 4x$
$= 6x + 7.$

From the foregoing instances the following theorems are easily inferred:—

(55). A constant which enters a function by addition or subtraction disappears in the difference. Hence the differences of functions that differ from each other by a constant are equal. Thus,

$$\Delta(u_x \pm a) = \Delta u_x.$$

(56). A constant entering a function as a multiplier appears in the same character in the difference of that function. Thus,

$$\Delta a u_x = a \Delta u_x.$$

(57). A compound function being an aggregation by means of the signs *plus* and *minus* of two or more simple functions, its difference will be a similar aggregation of the differences of the component functions. Thus, in the third example, u_x is the sum of two functions, $3x^2$ and $4x-6$, and we see that there $\Delta u_x = 3\Delta x^2 + 4\Delta x$. Generally,

$$\Delta(u_x \pm v_x \pm \ldots) = \Delta u_x \pm \Delta v_x \pm \ldots$$

(58). The operation symbolised by Δ is obviously one that admits of repetition. The result of its performance on Δu_x is denoted by $\Delta(\Delta u_x)$, which is abbreviated into $\Delta^2 u_x$. And generally the result of its performance n times is denoted by $\Delta^n u_x$. These results are called the first, second, ... nth differences of u_x, respectively.

(59). In the examples in (54) we saw that in each case the difference was one degree lower than the function from which it was derived; and I shall now show that in algebraical and integer functions this property holds universally. A consequence of this is that if u_x be a function of n dimensions, one dimension being lost at each operation, $\Delta^n u_x$ will be a constant, and differences of higher orders will be equal to nothing.

(60). Let $u_x = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots.$

$$\therefore\ u_{x+1} = A(x+1)^n + B(x+1)^{n-1} + C(x+1)^{n-2} + \ldots$$

Expanding the terms of u_{x+1}, we get,

$$\begin{array}{ll} A(x+1)^n &= Ax^n + Anx^{n-1} + \dfrac{An(n-1)}{2}x^{n-2} + \ldots \\ B(x+1)^{n-1} &= \qquad\quad Bx^{n-1} + B(n-1)x^{n-2} + \ldots \\ C(x+1)^{n-2} &= \qquad\qquad\qquad\qquad\quad Cx^{n-2} + \ldots \\ \vdots & \qquad\qquad\qquad\qquad\qquad\quad \vdots \end{array}$$

Hence,

$$u_{x+1} = Ax^n + (An+B)x^{n-1} + \left\{\left(\frac{An}{2} + B\right)(n-1) + C\right\}x^{n-2} + \ldots$$

And from this subtracting u_x we get

$$\Delta u_x = Anx^{n-1} + \left(\frac{An}{2} + B\right)(n-1)x^{n-2} + \ldots$$

$$= Anx^{n-1} + B_1x^{n-2} + C_1x^{n-3} + \ldots$$

putting B_1, C_1, &c., for the coefficients of x^{n-2}, x^{n-3}, &c.

Performing a similar operation on this expression for Δu_x, we should find

$$\Delta^2 u_x = An(n-1)x^{n-2} + B_2x^{n-3} + C_2x^{n-4} + \ldots$$

in which B_2, C_2, &c., are new coefficients.

Another performance of the operation upon this would give, in like manner,

$$\Delta^3 u_x = An(n-1)(n-2)x^{n-3} + B_3x^{n-4} + \ldots;$$

and hence we may infer generally, that the result of the nth performance of the operation Δ would be

$$\Delta^n u_x = An(n-1)(n-2)\ldots(n-n+1)x^{n-n},$$

which is evidently constant, and may be written

$$\Delta^n u_x = 1.2.3\ldots n.A.$$

(61). Examples of the theorem just established are—

$$\Delta^2 x^2 = 1.2 = 2; \quad \Delta^3 x^3 = 1.2.3 = 6; \quad \Delta^4 x^4 = 1.2.3.4 = 24.$$

In all these examples $A=1$. If $u_x = 2x^3 + 7x + 6$, we have $\Delta^3 u_x = 1.2.3.2 = 12$, A being here $=2$.

(62). In forming the successive differences of a function, Horner's transforming process becomes immediately available.

Let it be required to form the successive differences of

$$u_x=2x^3-6x^2+5x-12.$$

The entire process is in the margin. Transforming u_x into u_{x+1} (13), we have for the coefficients of the latter function, 2, 0, -1, -11, from which subtracting, as they stand, at the tops of the several columns, the coefficients of u_x, we have for those of Δu_x, 6, -6, 1. Treating these in the same way, we get 12, 0, for the coefficients of $\Delta^2 u_x$; and a repetition of the operation on these gives 12 for the coefficient of $\Delta^3 u_x$. We thus have

$$\begin{array}{ccccl}
2 & -6 & 5 & -12 & (1 \\
 & -4 & 1 & -11 & \\
 & -2 & -1 & & \\
 & 0 & & & \\
\hline
 & 6 & -6 & 1 & (1 \\
 & & 0 & 1 & \\
 & & 6 & & \\
\hline
 & & 12 & 0 & (1 \\
 & & & 12 & \\
 & & & \overline{12} &
\end{array}$$

$$\begin{aligned}
u_x &= 2x^3-6x^2+5x-12 \\
\Delta u_x &= 6x^2-6x+1 \\
\Delta^2 u_x &= 12x+0 \\
\Delta^3 u_x &= 12. \quad (60).
\end{aligned}$$

(63). The successive differences of a function may also be found in another way.

Since, (52),

$$\Delta u_x=u_{x+1}-u_x,$$

therefore, differencing both sides, (57),

$$\Delta^2 u_x=\Delta u_{x+1}-\Delta u_x.$$

In like manner,

$$\Delta^3 u_x=\Delta^2 u_{x+1}-\Delta^2 u_x;$$

and, generally,

$$\Delta^n u_x=\Delta^{n-1} u_{x+1}-\Delta^{n-1} u_x.$$

From this it appears that a difference of any order, may be formed from two adjoining differences of the next lower order, in a manner analogous to that in which the first difference is formed from two successive values of the function.

(64). The property just proved suggests the following process:—

u	Δ^1	Δ^2	Δ^3	Δ^4
$u_x=2x^3-6x^2+5x-12$	$6x^2-6x+1$	$12x+0$	12	0
$u_{x+1}=2x^3+0x^2-x-11$	$6x^2+6x+1$	$12x+12$	12	
$u_{x+2}=2x^3+6x^2+5x-10$	$6x^2+18x+13$	$12x+24$	⋮	
$u_{x+3}=2x^3+12x^2+23x+3$	$6x^2+30x+37$	⋮	⋮	
$u_{x+4}=2x^3+18x^2+53x+40$	⋮	⋮	⋮	
⋮	⋮	⋮	⋮	

The first column here contains five successive values of u_x, all formed from the first of them by successive changes of x into $x+1$, (13). The remaining columns are formed, each from that which precedes it, by subtracting each value in the latter from that which follows it. The second column consequently contains Δu_x, Δu_{x+1}, &c.; the third, $\Delta^2 u_x$, $\Delta^2 u_{x+1}$, &c.; and the fourth, $\Delta^3 u_x$, $\Delta^3 u_{x+1}$, &c., which last are equal to each other, the function being of three dimensions, (60). By the arrangement here adopted (which is not that most commonly employed) each function and its successive differences are found in the same horizontal line, which is very convenient.

(65). It is at once apparent that the process just exemplified is much more laborious than the one for the same purpose exemplified in (62). That last mentioned, therefore, will be generally preferred when the function to be differenced is given in symbols. It is otherwise, however, when the data are successive numerical values of a function. In this case the process of (64) alone is applicable; and in this case too it loses its irksomeness.

u	Δ^1	Δ^2	Δ^3	Δ^4
1438	541	120	12	0
1979	661	132	12	
2640	793	144	⋮	
3433	937	⋮	⋮	
4370	⋮	⋮	⋮	
⋮	⋮	⋮	⋮	

Above is a numerical example. The function is the same as before, the successive values of x here being 10 to 14. And the differences formed in (62) and (64) will be found to be the same as those formed above, when the proper values of the variable are substituted in them.

(66). To express $\Delta^n u_x$ in terms of u_x and succeeding values.

Since

$$\Delta u_x = u_{x+1} - u_x,$$

$$\therefore\ \Delta^2 u_x = \Delta u_{x+1} - \Delta u_x = (u_{x+2} - u_{x+1}) - (u_{x+1} - u_x)$$

$$= u_{x+2} - 2u_{x+1} + u_x.$$

Similarly,

$$\Delta^3 u_x = \Delta u_{x+2} - 2\Delta u_{x+1} + \Delta u_x$$

$$= (u_{x+3} - u_{x+2}) - 2(u_{x+2} - u_{x+1}) + (u_{x+1} - u_x)$$

$$= u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x.$$

The coefficients are obviously those of the expansion of $(1-1)^n$, so that we should finally have

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{2} u_{x+n-2} - \frac{n(n-1)(n-2)}{2.3} u_{x+n-3} + \dots . \text{*}(-1)^n u_x.$$

(67). The number of terms in this expression is $n+1$; so that, to enable us to form a specific difference of a function, we require to be furnished with one value more of the function than the number that denotes the order of the differences to be formed. Thus, to form a second difference, three values are necessary; a third difference, four values, and so on.

(68). The following is a particular application of the theorem in (66):—

If $u_x = x^m$, then will $u_{x+n} = (x+n)^m$, $u_{x+n-1} = (x+n-1)^m$, and so on. Therefore, by the theorem,

$$\Delta^n x^m = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{2}(x+n-2)^m - \dots . (-1)^{n+1} x^m.$$

Now, let $x=0$; that is, let the successive values of the function be 0^m, 1^m, 2^m, ... and the above becomes

$$\Delta^n 0^m = n^m - n(n-1)^m + n\frac{n-1}{2}(n-2)^m - n\frac{(n-1)(n-2)}{2.3}(n-3)^m \dots . \dots . (-1)^{n+1} n.1^m.$$

(69). The numbers expressed by this formula, by giving to n and m different values, are called *The Differences of* 0^m; or, more briefly, *Differences of Nothing*. They are of frequent occurrence, and of considerable use in analysis; and as we shall have occasion for them hereafter, I insert here a table of their values as far as $\Delta^6 0^6$.

Differences of Nothing.

0^m	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0^1	1	0	0	0	0	0
0^2	1	2	0	0	0	0
0^3	1	6	6	0	0	0
0^4	1	14	36	24	0	0
0^5	1	30	150	240	120	0
0^6	1	62	540	1560	1800	720

The table may be verified as in the margin, where the formation of the fourth line is shown.

$0^4 = 0$	1	14	36	24
$1^4 = 1$	15	50	60	
$2^4 = 16$	65	110		
$3^4 = 81$	175			
$4^4 = 256$				

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* The symbol $(-1)^n$ is a convenient substitute for the periphrasis, "*plus* or *minus* according as n is even or odd." And, in like manner, $(-1)^{n+1}$ expresses, "*plus* or *minus* according as n is odd or even."

We know already, (60), that when $n=m$, $\Delta^n 0^m=1.2.3\ldots n$; from which it follows that when n is greater than m, $\Delta^n 0^m=0$. These properties are visible from the table.

(70.) To express u_{x+n} in terms of u_x and its successive differences.

Since, (52),

$$u_{x+1}=u_x+\Delta u_x,$$

adding to each side its difference, this gives

$$u_{x+1}+\Delta u_{x+1}=u_x+\Delta u_x+\Delta(u_x+\Delta u_x);$$

or,

$$u_{x+2}=u_x+2\Delta u_x+\Delta^2 u_x.$$

Similarly,

$$u_{x+3}=u_x+3\Delta u_x+3\Delta^2 u_x+\Delta^3 u_x.$$

The binomial coefficients we see again present themselves, and therefore we conclude generally, that

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^3 u_x\ldots\ldots+\Delta^n u_x.$$

(71). In the expression just deduced n is the variable, and u_x, Δu_u, are constant. We may, therefore, for these, write u_0, Δu_0, &c., respectively. We thus get

$$u_n=u_0+n\Delta u_0+\frac{n(n-1)}{2}\Delta^2 u_0+\ldots,$$

in which we may, if we please, write x for n, both symbols being equally indeterminate.

(72). The expression in (70) enables us, when the requisite number of successive numerical values of a function is given, to determine the function. Thus, multiplying out the coefficients, arranging in powers of n, and going as far as the fifth order of differences, we get (writing, for brevity, Δ^1, Δ^2, &c., for $\Delta^1 u_x$, $\Delta^2 u_x$, &c.)

$$\begin{aligned}u_{x+n}=u_x&+\left(\Delta^1-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\frac{\Delta^5}{5}-\ldots\right)n\\&+\left(\frac{\Delta^2}{2}-\frac{\Delta^3}{2}+\frac{11\Delta^4}{24}-\frac{5\Delta^5}{12}+\ldots\right)n^2\\&+\left(\frac{\Delta^3}{6}-\frac{\Delta^4}{4}+\frac{7\Delta^5}{24}-\ldots\right)n^3+\left(\frac{\Delta^4}{24}-\frac{\Delta^5}{12}+\ldots\right)n^4\\&+\left(\frac{\Delta^5}{120}-\ldots\right)n^5+\ldots.\end{aligned}$$

We thus have, u_x, $\Delta^1 u_x$, &c., being known, a finite expression in terms of n, for u_{x+n}, from which it is easy to pass to u_x.

(73). As an example, u_{62}, u_{63}, &c., as in the following scheme, are given, and u_x is required.

x	u	Δ^1	Δ^2	Δ^3	
62	238328	11719	378	6	0
63	250047	12097	384	6	
64	262144	12481	390		
65	274625	12871			
66	287496				

Here $u_{62}=238328$, $\Delta u_{62}=11719$, $\Delta^2 u_{62}=378$, $\Delta^3=6$, $\Delta^4=0$, &c. Hence, substituting in (72),

$$\begin{aligned} u_{62+n} &= 238328+11532n+186n^2+n^3 \\ &= n^3+186n^2+11532n+238328. \end{aligned}$$

Now this will be transformed into $u_{62+n-62}=u_n$, as follows :—

$$\begin{array}{rrrrl} 1 & 186 & 11532 & 238328 & (-62 \\ & 126 & 3972 & 8^* & \\ & 66 & 12^* & 0 & \\ & 6^* & 4 & & \\ & 4 & 0 & & \\ & 2 & & & \\ & 0 & & & \end{array}$$

The result is n^3, all the coefficients, except the first, having disappeared. Hence, writing x for n, $u_x=x^3$; and, consequently, the given values are the cubes of 62 and following numbers.

(74). It is worth noting, in connexion with the foregoing transformation, that the absolute term, 238 . . . , being exhausted, —62, the transforming number, is a root of the polynomial operated upon (19). But the two preceding coefficients are also exhausted by the same transformation. This shows that there are other two roots, each equal to —62; and hence the polynomial can be nothing else than the expansion of $(n+62)^3$, which on trial will be found to be the case.

(75). As another example, a certain function, u_x, when x is 20, 21, &c., assumes the values shown below. Required the function.

x	u	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	
20	3047619	850841	180260	27240	2616	120	0
21	3898460	1031101	207500	29856	2736	120	
22	4929561	1238601	237356	32592	2856		
23	6168162	1475957	269948	35448			
24	7644119	1745905	305396				
25	9390024	2051301					
26	11441325						

Here $u_{20}=3047619$, $\Delta u_{20}=850841$, $\Delta^2 u_{20}=180260$, $\Delta^3 u_{20}=27240$,

$\Delta^4 u_{20}=2616$, $\Delta^5 u_{20}=120$, $\Delta^6 u_{20}=0$, &c. Hence, substituting these values in (72), we get

$$u_{20+n}=n^5+99n^4+3921n^3+77659n^2+769161n+3047619.$$

Now transform as below—

1	99	3921	77659	769161	3047619(−20
	79	2341	30839	152381	−1
	59	1161	7619	1	
	39	381	−1		
	19	1			
	−1				

And the function is

$$u_x=x^5-x^4+x^3-x^2+x-1.$$

(76). Resuming the expression in (70),

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{2}\Delta^2 u_x+\ldots\ldots;$$

by giving n in this expression the integer values in succession, 0, 1, 2, &c., we of course reproduce the terms u_x, u_{x+1}, u_{+2}, &c., from which the differences Δu_x, $\Delta^2 u_x$, &c., were deduced. But also, if we give to n fractional values, we should produce terms intermediate to those just referred to; and we shall thus be able, by employing proper values, to divide the intervals u_x to u_{x+1}, u_{x+1} to u_{x+2}, &c., into any required number of parts. Thus, if we make $n=\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, &c., successively, we shall form the values of $u_{x+1:2}$, u_{x+1}, $u_{x+3:2}$, and so on, and thus the original intervals will be bisected. Let, generally, $n=\frac{t}{m}$, and the expression will become

$$u_{x+t:m}=u_x+\frac{t}{m}\Delta u_x+\frac{\frac{t}{m}\left(\frac{t}{m}-1\right)}{2}\Delta^2 u_x+\frac{\frac{t}{m}\left(\frac{t}{m}-1\right)\left(\frac{t}{m}-2\right)}{2.3}\Delta^3 u_x+\ldots$$

$$=u_x+t\cdot\frac{\Delta u_x}{m}+\frac{t(t-m)}{2}\cdot\frac{\Delta^2 u_x}{m^2}+\frac{t(t-m)(t-2m)}{2.3}\cdot\frac{\Delta^3 u_x}{m^3}+\ldots$$

On multiplying out the coefficients, and going as far as the sixth order of differences, this becomes

$$u_{x+t:m}=u_x+t\cdot\frac{\Delta u_x}{m}+\frac{t^2-mt}{2}\cdot\frac{\Delta^2 u_x}{m^2}+\frac{t^3-3mt^2+2m^2t}{6}\cdot\frac{\Delta^3 u_x}{m^3}$$

$$+\frac{t^4-6mt^3+11m^2t^2-6m^3t}{24}\cdot\frac{\Delta^4 u_x}{m^4}$$

$$+\frac{t^5-10mt^4+35m^2t^3-50m^3t^2+24m^4t}{120}\cdot\frac{\Delta^5 u_x}{m^5}$$

$$+\frac{t^6-15mt^5+85m^2t^4-225m^3t^3+274m^4t^2-120m^5t}{720}\cdot\frac{\Delta^6 u_x}{m^6}.$$

(77.) Now, to form the differences of this expression, taking the nth difference of both sides with respect to t (remembering that u_x, Δu_x, &c., are constant), making then $t=0$, and writing $\delta^n u_x$ on the left-hand side of the equation, to remind us that the difference has reference to the interval $1:m$, we get

$$\begin{aligned}
\delta^n u_x = {} & \frac{\Delta^n 0^1}{1} \cdot \frac{\Delta u_x}{m} + \frac{\Delta^n 0^2 - m\Delta^n 0^1}{2} \cdot \frac{\Delta^2 u_x}{m^2} \\
& + \frac{\Delta^n 0^3 - 3m\Delta^n 0^2 + 2m^2\Delta^n 0^1}{6} \cdot \frac{\Delta^3 u_x}{m^3} \\
& + \frac{\Delta^n 0^4 - 6m\Delta^n 0^3 + 11m^2\Delta^n 0^2 - 6m^3\Delta^n 0^1}{24} \cdot \frac{\Delta^4 u_x}{m^4} \\
& + \frac{\Delta^n 0^5 - 10m\Delta^n 0^4 + 35m^2\Delta^n 0^3 - 50m^3\Delta^n 0^2 + 24m^4\Delta^n 0^1}{120} \cdot \frac{\Delta^5 u_x}{m^5} \\
& + \frac{\Delta^n 0^6 - 15m\Delta^n 0^5 + 85m^2\Delta^n 0^4 - 225m^3\Delta^n 0^3 + 274m^4\Delta^n 0^2 - 120m^5\Delta^n 0^1}{720} \cdot \frac{\Delta^6 u_x}{m^6}.
\end{aligned}$$

From this we get, making $n=1$, 2, &c., successively, and taking the values of the coefficients from the table of the Differences of Nothing in (69), the following expressions for $\delta^1 u_x$, $\delta^2 u_x$, &c.:—

$$\begin{aligned}
\delta^1 u_x = {} & \frac{\Delta u_x}{m} + \frac{1-m}{2} \cdot \frac{\Delta^2 u_x}{m^2} + \frac{1-3m+2m^2}{6} \cdot \frac{\Delta^3 u_x}{m^3} \\
& + \frac{1-6m+11m^2-6m^3}{24} \cdot \frac{\Delta^4 u_x}{m^4} \\
& + \frac{1-10m+35m^2-50m^3+24m^4}{120} \cdot \frac{\Delta^5 u_x}{m^5} \\
& + \frac{1-15m+85m^2-225m^3+274m^4-120m^5}{720} \cdot \frac{\Delta^6 u_x}{m^6}.
\end{aligned}$$

$$\begin{aligned}
\delta^2 u_x = {} & \frac{\Delta^2 u_x}{m^2} + (1-m)\frac{\Delta^3 u_x}{m^3} + \frac{7-18m+11m^2}{12} \cdot \frac{\Delta^4 u_x}{m^4} \\
& + \frac{3-14m+21m^2-10m^3}{12} \cdot \frac{\Delta^5 u_x}{m^5} \\
& + \frac{31-225m+595m^2-675m^3+274m^4}{360} \cdot \frac{\Delta^6 u_x}{m^6}.
\end{aligned}$$

$$\begin{aligned}
\delta^3 u_x = {} & \frac{\Delta^3 u_x}{m^3} + \frac{3(1-m)}{2} \cdot \frac{\Delta^4 u_x}{m^4} + \frac{5-12m+7m^2}{4} \cdot \frac{\Delta^5 u_x}{m^5} \\
& + \frac{6-25m+34m^2-15m^3}{8} \cdot \frac{\Delta^6 u_x}{m^6}.
\end{aligned}$$

$$\delta^4 u_x = \frac{\Delta^4 u_x}{m^4} + 2(1-m)\frac{\Delta^5 u_x}{m^5} + \frac{13-30m+17m^2}{6} \cdot \frac{\Delta^6 u_x}{m^6}.$$

$$\delta^5 u_x = \frac{\Delta^5 u_x}{m^5} + \frac{5(1-m)}{2}\cdot\frac{\Delta^6 u_x}{m^6}.$$

$$\delta^6 u_x = \frac{\Delta^6 u_x}{m^6}.$$

(78). The foregoing expressions are easily transformed into the following more convenient forms:—

$$\begin{aligned}\delta^1 u_x = {} & \frac{\Delta u_x}{m} - \frac{m-1}{2}\cdot\frac{\Delta^2 u_x}{m^2} + \frac{(m-1)(2m-1)}{6}\cdot\frac{\Delta^3 u_x}{m^3}\\ & - \frac{(m-1)(2m-1)(3m-1)}{24}\cdot\frac{\Delta^4 u_x}{m^4}\\ & + \frac{(m-1)(2m-1)(3m-1)(4m-1)}{120}\cdot\frac{\Delta^5 u_x}{m^5}\\ & - \frac{(m-1)(2m-1)(3m-1)(4m-1)(5m-1)}{720}\cdot\frac{\Delta^6 u_x}{m^6}.\end{aligned}$$

$$\begin{aligned}\delta^2 u_x = {} & \frac{\Delta^2 u_x}{m^2} - (m-1)\frac{\Delta^3 u_x}{m^3} + \frac{(m-1)(11m-7)}{12}\cdot\frac{\Delta^4 u_x}{m^4}\\ & - \frac{(m-1)(2m-1)(5m-3)}{12}\cdot\frac{\Delta^5 u_x}{m^5}\\ & + \frac{(m-1)(2m-1)(137m^2-132m+31)}{360}\cdot\frac{\Delta^6 u_x}{m^6}.\end{aligned}$$

$$\begin{aligned}\delta^3 u_x = {} & \frac{\Delta^3 u_x}{m^3} - \frac{3(m-1)}{2}\cdot\frac{\Delta^4 u_x}{m^4} + \frac{(m-1)(7m-5)}{4}\cdot\frac{\Delta^5 u_x}{m^5}\\ & - \frac{(m-1)(3m-2)(5m-3)}{8}\cdot\frac{\Delta^6 u_x}{m^6}.\end{aligned}$$

$$\delta^4 u_x = \frac{\Delta^4 u_x}{m^4} - 2(m-1)\frac{\Delta^5 u_x}{m^5} + \frac{(m-1)(17m-13)}{6}\cdot\frac{\Delta^6 u_x}{m^6}.$$

$$\delta^5 u_x = \frac{\Delta^5 u_x}{m^5} - \frac{5(m-1)}{2}\cdot\frac{\Delta^6 u_x}{m^6}.$$

$$\delta^6 u_x = \frac{\Delta^6 u_x}{m^6}.$$

(79). The results just deduced are of great importance. They enable us, when the requisite number of equidistant terms of a series is known—and that whether the characteristic function be algebraical or transcendental—to interpolate the intermediate terms. We shall have frequent occasion for their application in the fourth section.

(80). They have a somewhat formidable appearance. But it must be noted, first, that it is but very rarely that it is required to use so many as six orders of differences; so that, when fewer suffice, the terms involving the higher orders of the differences of

u_x vanish. And, secondly, that when the numerical value of m (which is one more than the number of terms to be interpolated in each interval) is inserted, the expressions become materially simplified.

(81). The most generally convenient value of m is 10. That is, it is usually arranged that the given—the fundamental or primitive—terms should be ten apart in the series to be formed. In this case the numerical values of the formulæ in (78) are as follows:—

$$\delta^1 u_x = \frac{\Delta^1 u_x}{10} - \frac{4{\cdot}5\Delta^2 u_x}{10^2} + \frac{28{\cdot}5\Delta^3 u_x}{10^3} - \frac{206{\cdot}625\Delta^4 u_x}{10^4} + \frac{1611{\cdot}675\Delta^5 u_x}{10^5} - \frac{13162{\cdot}0125\Delta^6 u_x}{10^6},$$

$$\delta^2 u_x = \;.\;.\; \frac{\Delta^2 u_x}{10^2} - \frac{9\Delta^3 u_x}{10^3} + \frac{77{\cdot}25\Delta^4 u_x}{10^4} - \frac{669{\cdot}75\Delta^5 u_x}{10^5} + \frac{5895{\cdot}225\Delta^6 u_x}{10^6}.$$

$$\delta^3 u_x = \;.\;.\;.\;.\;.\;.\; \frac{\Delta^3 u_x}{10^3} - \frac{13{\cdot}5\Delta^4 u_x}{10^4} + \frac{146{\cdot}25\Delta^5 u_x}{10^5} - \frac{1480{\cdot}5\Delta^6 u_x}{10^6}.$$

$$\delta^4 u_x = \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; \frac{\Delta^4 u_x}{10^4} - \frac{18\Delta^5 u_x}{10^5} + \frac{235{\cdot}5\Delta^6 u_x}{10^6}.$$

$$\delta^5 u_x = \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; \frac{\Delta^5 u_x}{10^5} - \frac{22{\cdot}5\Delta^6 u_x}{10^6}.$$

$$\delta^6 u_x = \;.\; \frac{\Delta^6 u_x}{10^6}.$$

(82). The differences are so arranged in the above scheme, that by drawing, or conceiving to be drawn, a vertical line at the proper place, the terms of the several series which are not wanted in any particular application are cut off. Thus, if two orders of differences suffice, we have for the working formulæ—

$$\delta^1 u_x = \frac{\Delta^1 u_x}{10} - \frac{4{\cdot}5\Delta^2 u_x}{10^2}, \text{ and } \delta^2 u_x = \frac{\Delta^2 u_x}{10^2};$$

if three orders are needed, we have

$$\delta^1 u_x = \frac{\Delta^1 u_x}{10} - \frac{4{\cdot}5\Delta^2 u_x}{10^2} + \frac{28{\cdot}5\Delta^3 u_x}{10^3},$$

$$\delta^2 u_x = \;.\;.\; \frac{\Delta^2 u_x}{10^2} - \frac{9\Delta^3 u_x}{10^3},$$

$$\delta^3 u_x = \;.\;.\;.\;.\;.\;.\; \frac{\Delta^3 u_x}{10^3};$$

and so on.

(83). Another useful value of m is $m=5$. It may be well, therefore, for facilitation of reference, to give here the results of the substitution of this value in the formulæ of (78).

$$\begin{aligned}
\delta^1 u_x &= \frac{\Delta^1 u_x}{5} - \frac{2\Delta^2 u_x}{5^2} + \frac{6\Delta^3 u_x}{5^3} - \frac{21\Delta^4 u_x}{5^4} + \frac{79{\cdot}8\Delta^5 u_x}{5^5},\\
\delta^2 u_x &= \quad . \; . \quad \frac{\Delta^2 u_x}{5^2} - \frac{4\Delta^3 u_x}{5^3} + \frac{16\Delta^4 u_x}{5^4} - \frac{66\Delta^5 u_x}{5^5},\\
\delta^3 u_x &= \quad . \; . \; . \; . \; . \quad \frac{\Delta^3 u_x}{5^3} - \frac{6\Delta^4 u_x}{5^4} + \frac{30\Delta^5 u_x}{5^5},\\
\delta^4 u_x &= \quad . \; . \; . \; . \; . \; . \; . \; . \quad \frac{\Delta^4 u_x}{5^4} - \frac{8\Delta^5 u_x}{5^5},\\
\delta^5 u_x &= \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \frac{\Delta^5 u_x}{5^5}.
\end{aligned}$$

(84). These will, perhaps, be rather more easily used if put in the following form:—

$$\begin{aligned}
\delta^1 u_x &= 2\frac{\Delta^1 u_x}{10} - 8\frac{\Delta^2 u_x}{10^2} + 48\frac{\Delta^3 u_x}{10^3} - 336\frac{\Delta^4 u_x}{10^4} + 2553{\cdot}6\frac{\Delta^5 u_x}{10^5},\\
\delta^2 u_x &= \quad . \; . \quad 4\frac{\Delta^2 u_x}{10^2} - 32\frac{\Delta^3 u_x}{10^3} + 256\frac{\Delta^4 u_x}{10^4} - 2112\frac{\Delta^5 u_x}{10^5},\\
\delta^3 u_x &= \quad . \; . \; . \; . \; . \quad 8\frac{\Delta^3 u_x}{10^3} - 96\frac{\Delta^4 u_x}{10^4} + 960\frac{\Delta^5 u_x}{10^5},\\
\delta^4 u_x &= \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad 16\frac{\Delta^4 u_x}{10^4} - 256\frac{\Delta^5 u_x}{10^5},\\
\delta^5 u_x &= \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad 32\frac{\Delta^5 u_x}{10^5}.
\end{aligned}$$

General Average. *By* RICHARD MORRISON.

(Concluded from p. 51.)

IN the Report before alluded to, this case is especially noticed as affording evidence of the concessions made by foreign countries to obtain uniformity. The practice abroad has hitherto conformed to the principle recognized universally, as to the propriety of admitting as general average the damage done by voluntary stranding which was attended by the preservation of the ship and cargo. They still retain, however, as a subject of contribution, any damage which may arise from running a ship ashore to avoid capture; but it is clear that the same objection cannot be raised as in the case of a vessel in danger of foundering; for she may be, under the former circumstances, perfectly seaworthy, and therefore there would not be the difficulty of determining the exact amount of damage done to the cargo or the ship.

By the law of the United States the cargo saved contributes for the loss incurred by the ship and freight, even if the ship should be totally lost, when voluntarily stranded for the general benefit.* Stevens states, that from manuscript statements in his possession, it appears to have been the practice of Lloyd's to allow the damage in the time of Weskett (that is, about the year 1781), in common with foreign countries, and that it was derived from the Rhodian law. He thought the practice "unreasonable and unfounded," but cites the opinion of Marshall, who considered the damage sustained and the expenses incurred to be a fit case for contribution where the ship was got off and subsequently repaired. This exactly accords with the rule first proposed and embodied in the Draft Bill which formed the basis of the discussion in 1864. It was there suggested that such damage to the ship or cargo should be allowed as general average, if the act took place to avoid capture or foundering, if she was afterwards got off and repaired; but if, when she was got off, it was found impossible to repair her, or the expense of doing so would be so great as to exceed her value, then no compensation was to be allowed.

We are inclined to think that the adoption of this rule would be better calculated to obtain entire uniformity than the one finally decided upon; as the modification, being a middle course, would rather meet with general approval than a complete alteration in the practice; for foreigners, by *conceding* the point, do not acknowledge it to be unjust or wrong in principle, and therefore in practice would be unwilling to abandon it. It will have been seen that in the question of wages and provisions of the crew, a middle course was proposed. Foreign nations admitted them without qualification. We rejected them; but now propose to admit them in such a manner that, by the greater portion of them being allowed, and both parties consenting to such an arrangement, few complaints of injustice can be made.

We now come to the apportionment of the general average, and proceed to state the values of the interests to contribute. These interests have been frequently mentioned in the course of these remarks, as ship, freight, and cargo. But each of them has a different value attached to it at various periods during the adventure by which they are all associated. The ship has a different value to her owner when she arrives at her port of destination, after battling with winds and waves, to that which she had when she left her port of departure all "tight, staunch, firm and strong;" much more then, when meeting with more than ordinary perils of the sea she is compelled to put into a port of refuge. Another consideration in estimating the value of the ship, arises from the fact that her owner may perceive beauties in her, which cause him to set a higher value upon her than would be perhaps applied if the ordinary test were taken, viz., her strength, build, and rig. He may imagine, or know, that she possesses greater powers of sailing and facilities for the carriage of certain cargoes, which to common observers are imperceptible, but which may cause him to estimate her worth at an amount quite different from that she would realize if sold in a market, or even different from her prime cost. Then the freight, which if paid to the shipowner or charterer at the commencement of the voyage, might be a large sum, gradually diminishes as the voyage progresses; for he has to pay and support the crew out of it, so that each day the value is less to him than at the outset of the adventure. The cargo likewise has a certain value at the port of shipment, which may be greater or less on the arrival of the vessel at her destination. It is supposed by the merchant to be greater, but the markets may fluctuate and the result not correspond with his anticipations: the merchandize may produce very little or realize an extraordinary sum, according to the demand and supply or other circumstances connected with the subject-matter of the sale.

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* For an account of the case which decided the question, and for the opinions held on this point, by legislators and jurists, see Arnould, pp. 900–902.

Which value, then, is to be taken? At what period of the voyage can a fair estimate be made, so that all the valuations will agree, taking them as nearly as possible upon one basis; for that is the equitable mode of procedure, and the one sought after? I shall endeavour to show at some length, that, according to the method usually adopted in estimating the value of the ship, in *principle* the *correct* value is not taken, but that *practically* it answers the same end; thus affording another illustration how little agreement sometimes exists between principle and practice when applied to the adjustment of general average losses.

This is confessedly one of the greatest difficulties connected with the subject. With regard to the contributory interests generally, different States have adopted various methods, and in England the law has varied considerably at different periods. Stevens, quoting the words of Abbott, "a learned and very useful writer," says, "There is no principle of maritime law that has been followed by more variations in practice than this; and the determinations of English courts of justice furnish less of authority on this subject than on any other branch of maritime law."

It formed one of the questions which recent discussions have endeavoured to set at rest, and the result has been to admit the propriety of the mode of valuation adopted in England. We wish to call attention to the words of the section, that "the contribution shall be made upon the *actual* values of the property at the termination of the adventure." This, it will be seen, agrees with the opinions, hereafter adduced, of all who have given the matter their consideration.

Stevens—whom we have alluded to as the first writer on average, and whose work is continually quoted by every writer on marine insurance and average—says, "Whatever be the nature of the property, it is the value at *risk* (*i.e.*, the value to the owner on arrival) that should contribute, and not any former or after value of the thing." The cargo, on its "true value saved"; and for the ship, "the true value for contribution is, the amount that her hull, masts, yards, sails, rigging and stores, would produce after the sacrifice is made—with the addition of the amount made good by the general average contribution"; or in the words of Marshall, which he gives, "the ship is to be valued at the price she is worth on her arrival at her port of delivery. Sir Joseph Arnould, one of the modern writers, gives the general principle of valuation in the words of another: "The contributory value of the different interests is their value to their owners at the time and place to which the apportionment relates"—that is, in most cases, the port of destination. He observes, that although this principle is simple, it has given rise to many difficulties in its practical application; and assigns as the chief reason, the fact that the difference between the values for sacrifice and for expenditures has not been sufficiently remembered. The distinction is this: in the case of *expenditures* contribution is due from the *moment of the outlay*, and is payable apart from all future contingencies that may arise in the further prosecution of the voyage. But with *sacrifices* the case is different, for the property at risk is not held to be liable for contribution till it has arrived at the place where the adjustment is made; which place should be the port of discharge, on the ship's arrival there. Sir J. Arnould, confirming the view of Mr. Phillips, the American writer, considers that where the loss adjusted has arisen partly from expenses and partly from sacrifices, "the contributory value of the property saved ought in theory to be estimated on two different principles"; but he truly remarks that such a course would be "attended with a degree of difficulty and embarrassment inconsistent with the exigencies of actual business." Such a course is never adopted.

Now, applying the general principle laid down by these authorities with regard to the valuations, that of the ship will be "the amount at which her owner could afford to sell her at the time and place at which the adjustment is made. Her contributory value, in fact, as the rule may be shortly given, *is her worth to the owners in the state in which she arrives.*"*

I have preferred giving the exact words of these authorities, because in examining whether the principle is acted upon as it is broadly laid down, by defining the terms it will be seen that the *actual value* of the ship, according to the proper meaning of the expression, by the method adopted in England and elsewhere, is not taken at all; the one taken being her worth at the time of the accident, computed from her value at the port of departure, minus the damage sustained. It is different in principle from the one stated above, and apparently dissimilar from that of the freight and cargo, although it *practically* amounts to nearly the same thing; and in countries so contiguous that the prices of shipping are the same, there can be no question raised as to its propriety.

What is the *value* of any thing? Is it—amplifying but adhering to the sense of the dictionary meanings—the mere rate of worth set upon it by general custom, according to the nature and class of the commodity; the amount for which it will sell, and allowing the possession of certain distinctive qualities to render it more valuable at one time than another, even when compared with objects of the same kind? Or, is it regulated by laws which admit of no exception in the specific class of merchandize or commodities to which it applies? "Value," says Mr. H. Fawcett, "implies the comparison of one commodity with another. It is, therefore, a relative expression." Mr. J. S. Mill, quoting Adam Smith, gives two meanings to the word—one signifying usefulness, the other, power of purchasing; and in reviewing these definitions, explains them by giving as a comprehensive meaning, "the command which the possession of anything gives over purchaseable commodities in general." He likewise distinguishes between *price* and *value*, terms which were used synonymously by the early economists. And he subsequently shows that the money value or price of a thing is, for general purposes, to be considered as equivalent to its exchange value or purchasing power, with the proviso that the prices of all other things remain unaltered, except the one under consideration.*

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* Arnould here refers to Stevens, Beneckė, and Phillips; and cites, in a note, a case decided in the United States, where a ship, after being saved by jettison, was so damaged that she was obliged to be sold, when the average was calculated on the price she sold for.—Vol. ii., p. 932, *et seq.*

We may infer, then, that the value or worth of anything will depend on its purchasing power, and this latter will be greatly influenced by its usefulness.† Now, in applying this proposition to the subject under consideration, namely, the value of the ship to contribute to general average, we may say that her *true* value cannot be that at the port of departure. "The contributory value of a ship is her pecuniary value at the time she becomes liable for the general average. . . . This, in most cases, is her value at that date to her owners, apart from any factitious value which may be imparted to her by contracts or other engagements." (Baily.)

Because a ship is *built for* and *valued at* so much per ton, is that value to be considered her true value at all times and in all places, and to vary only with the gradual rise or fall in the price of shipping generally, or the materials which produce it? As we shall see hereafter, the cargo is usually made to contribute on what it *actually* produces (less certain charges), and if not sold is estimated according to the market value of similar articles or produce. The question, therefore, arises, why, if the ship is equally interested in the distribution of the expenditures or losses, should a different valuation be taken for it? We set aside for the present the difficulty of arriving at the value of the ship in this way, merely regarding the question as one of principle; and remember likewise that there may be a market for goods where the value might be easily arrived at, whilst for shipping there might not be sufficient business done to warrant a fair price being assumed. This may be illustrated by circumstances which could easily arise. A vessel might be on a stated voyage. In consequence of a general average act she puts into a port of refuge, incurs expenses, after having sacrificed part of her cargo by jettison. The correct value for contribution is her value at that time to her owners—*i.e.*, the amount she would *realize* if sold at that port, according to the meaning of the term "*value*." In estimating this, the assessors ought, and probably would, take into account advantages which the vessel might possess for carrying on a trade from that port to another, from her draught, tonnage, or peculiarity of rig, which in some trades is of importance. This would be on the principle of her actual worth at the time the settlement was made between the respective interests associated.*

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* *Principles of Political Economy*, book iii., ch. 1.

† "Any article whatever, to obtain that artificial sort of value which is meant by exchange value, must begin by offering itself as a means to some desirable end."—De Quincey, quoted by Mill; *ibid.*, ch. 2.

The case would then stand thus:—the assessors would admit that at the port of departure the value of the vessel might be greater or less than the amount she would realize if she were to be sold at that port at the same date; but they would say, "it is needful to ascertain her actual value, and estimating it according to the rules commonly employed for determining the value of all things, we submit that she is worth such a sum, although she would not perhaps fetch it elsewhere. It is true there is no market for the ship; she was not sent out to be sold; but for the occasion we must suppose a market to exist, that supposition being necessary owing to the circumstances in which the ship is interested demanding that proceeding." They might further add, that the fact of its being a *maritime* market for goods, is a fair basis for the inference that vessels would be sometimes bought and sold, and therefore that—although at that particular date there was no demand for such a ship—a market value could be obtained without much difficulty, subject to the same conditions and fluctuations that apply to all things bought and sold. When freights are high, it shows that vessels are scarce and that there is a demand for them; or there may be plenty of vessels but not much demand, in which case freights would be low, and the value of shipping would decline, analogous to the laws which govern the prices of those commodities which constitute cargoes. The value of shipping depends also upon other circumstances. It might be more valuable in Calcutta or Sydney than in London or New York; but this might arise from the circumstances of the supply of timber, iron, and hemp, as well as from the scarcity or abundance of labour, or from the want of capital to undertake the building of vessels.† These facts are to be considered as applying rather to the relative standard of value which exists in various parts of the world, but which nevertheless remotely affect the economical value. "The demand partly depends on the value and the value depends on the demand." (J. S. Mill.)

* "The utility of a thing, in the estimation of a purchaser, is the extreme limit of its exchange value; higher the value cannot ascend; peculiar circumstances are required to raise it so high."—J. S. Mill, book iii., ch. 2.

"The price of a commodity which is actually realized may oscillate.... according as the vendor or purchaser has the more skill and knowledge of trading operations."—Fawcett's *Pol. Econ.*, p. 314.

† Ship-building has declined at Calcutta, but it has of late years risen to high importance at Moulmein, owing chiefly to the plentiful supply of teak, which wood is preferred above all others for such purposes.

This, then, is one view which we surmise might be taken in determining the value of the ship; let us now urge it a step further.

The equity of a principle is frequently tested by extreme cases. The object proposed is to show that to take what may be considered the true value of the ship at her port of destination, according to the strict meaning of the term, would not be so correct as that which is usually applied in the distribution of the general average.

Suppose a ship whose destination was Bombay, during the late American war, when such vast quantities of cotton were sent to England from India, meets with dangers which necessitate a general average act. At that time a great many vessels were employed which would not previously have obtained a cargo of that valuable description, owing to their inferior qualities. Suppose this ship to be worth upon leaving England, say £7 per ton; but owing to the great demand for vessels and the scarcity of the supply, would, if sold, realize £10 per ton; would not that be her value to her owners? And further, let us suppose that vessels become so scarce that £15 per ton might be demanded for the same ship. Ought that value to contribute to be set aside for another? Those who maintain that the actual value—what she would produce if sold—at the port of destination, is the correct one, we may imagine would reply in the negative; and assert that the excessive demand for shipping would result in the embarking of more capital in that branch of commercial enterprise, which would, in process of time, reduce the price to its normal condition; as it is a principle of commerce that the demand and supply regulate each other, and in the end become equalized.

On the other hand the advocates for the principle adopted might meet those who take this view on their own ground, and argue that as it is a law of all commercial relations that such a result must happen, and that eventually there is a tendency to the cost of production, it would be preferable to start from that point as a basis for estimating the value of the ship, as it is a merchantable interest, which has not a market everywhere like ordinary produce or merchandize. They might say that the value in the case supposed above was purely circumstantial and factitious, altogether an exceptional case, and that therefore it could not be cited as an illustration of a sound principle of commerce.

It will appear, then, that the price or value (using the terms synonymously) of a ship, like anything which forms the subject of commercial intercourse, may be referred back to its original cost of production, regarding that as "a position of stable equilibrium"; and that if this condition be interrupted, the ordinary laws are put into operation by the introduction of more capital into that branch of commerce which yields more than ordinary profits, so that the normal value soon results and average profits succeed. By this means the cost of production, including the profit, although affected by temporary fluctuations, may be assumed as a standard of value that might be applied to a vessel when such a valuation becomes necessary. This would approximate very nearly to the real value if sold at the port of destination, and may be considered equitable for general practice.

Many continental nations, including France, adopt the method of valuing the ship at her worth at the port of discharge; taking, however, only one-half of that value for contribution. Spain contributes upon the whole value, as if she were sold there and then. They also take half the value of the freight for contribution.

It must not be supposed that the writer wishes to prove that it is immaterial which value is taken, or that the two are so nearly equivalent that it is almost a question of terms. It is for the most part acknowledged that the values at the ports of departure and destination are different; the difficulty is to ascertain the actual one at the latter place—a matter almost impossible as a general rule, for the surveyors and nautical assessors could not properly estimate it without thoroughly examining the ship from stem to stern, and from the topmasts to the kelson; neither could they (unless she were in dry dock) obtain any knowledge of the state of the copper, the condition of which would inform them on many points connected with her value.

It may be affirmed that it is the exception to find a great increase in the value, beyond the average profit on the cost of building; and, as in giving the value to contribute the owner or master could take into account that increase at the port where the average was stated—should it exist and it were possible to arrive at it—no great practical injustice could arise.

But it may be objected—why not apply the same rule of "stable equilibrium" to the cargo; why not take the same value for it? To which it is answered, that when the average is adjusted at the port of departure it *is* taken: but it is clear that it cannot be so just as the value where the goods are actually sold, for the sale determines their true value or power of exchange to the owner of them. But another objection might be urged: according to this reasoning, is not the cost of production (including the charges and profit) considered equivalent to the sale value; do they not become equalized; why, therefore, is not the one as good as the other? It is quite true that they do so adjust themselves in process of time; but the law is, in England and in most other countries, as has been stated, that the *actual* value of the interests to the owners at the time the adjustment is made is the one that is to be applied. The values are subject to an increase or decrease in a certain ratio where regular markets exist; but as the ship is seldom sold unless she is unseaworthy, the same test cannot be universally employed, and it appears the best course to refer to the prime cost; although it must be remembered that the value at the port of departure, less the repairs required (or the *damage*) cannot be exactly the same as that which she would produce if sold at the port of destination, unless the market value of shipping and materials were the same in both places.

By English law the freight contributes on the net amount; that is, after deducting from its gross value, the wages of the master and seamen, port charges, and special charges where such have been incurred. This gives a result analogous to that obtained in the case of the ship. It is the amount actually saved or earned; all that the owner could receive at the termination of the voyage, of which it formed one of the associated interests. Where a vessel is in ballast, but under charter to earn freight, the freight she has commenced to earn contributes to general average.

The freight jettisoned with goods, which is recoverable as general average, is added to the contributory value of the freight, when it is payable on delivery of the goods. If any of the freight be advanced to the shipowner, that amount likewise contributes; it is an integral part of the whole gross freight; it is at risk to the merchant (who generally insures it), and whatever is actually at risk contributes to make good any loss which is the means of saving it, or which in any way benefits it.

It sometimes happens that the freight is consumed: it is, through the protraction of the voyage, swallowed up by the wages or by charges incident to it; in which case there is clearly nothing to contribute, for its owner has no value or gain by the transaction.

The value of the cargo for contribution is that which it actually produced at the port of delivery after deducting the freight, duty, landing and sale charges, and special charges where necessary. This will give the net market value.

Thus it appears that the interests over which the loss or expenditure is distributed are severally lessened, and put as nearly as possible on one footing, by the charges being subtracted from them.

The question has been sometimes raised as to what merchandize should contribute to general average; and as a proof of the antiquity of the principle of average, there have been cited letters patent, which Edward I. sent to the Cinque Ports in 1285, declaring what goods were to be held liable. As a general rule it may be said that all goods put on board for the purpose of trade must contribute. The ammunition, stores and provisions, do not; neither do the effects of passengers or crew, which consist of luggage, wearing apparel or jewellery. Gold and silver, either in bars or as specie, jewels and precious stones, contribute, if not included in the category of passengers' baggage. The old and concise rule is, "what pays no freight pays no average." However, many writers have held the opinion that articles of value which are proportionably benefited ought to contribute. It has in many cases happened that masters of vessels and passengers have carried goods as baggage, for which they paid no freight; so that as a matter of equity such goods ought to bear their proportion of the loss, for in effect they are part of the cargo. We are told that Weskett considered that bank notes ought to contribute; and his view is supported by Sir Joseph Arnould, on the ground that they are convertible into money, and being saved by the sacrifice are prevented from becoming valueless. Mr. Phillips takes a different view, maintaining that they are not so much property, as the evidence of property. If, however, we apply the old rule of Magens, just given, they ought not to contribute, for they pay no freight. By the old laws it was held that goods belonging to Government did not contribute; at the present day, however, no exception is made in their favour, and claims for general average are not unfrequently sent in to the store departments of Her Majesty's India and War Offices. The matter was settled in the United States in 1838, when it was decided that "there was no ground for their exemption either in law or practice."

Where a jettison has occurred, the merchandize so sacrificed is to be valued in proportion to the sound portion of the same description of cargo; but if all that remains of the cargo after the jettison should arrive in a damaged condition, then the value of that sacrificed is to be computed in proportion to the whole shipment; for it would not be a fair assumption that, where the whole shipment arrived in a damaged state, the goods jettisoned would be likely to have escaped sea damage. As the jettison would not be resorted to but in a great emergency, it would not be probable that the sea damage to the cargo arose altogether subsequent to the general average act.

That portion of the cargo which is jettisoned contributes to general average. It pays a proportion of its own loss, for it would be clearly unfair that the person whose merchandize was sacrificed should be the only one who did not lose, which would be the result if he did not contribute. The value of the cargo cast overboard is therefore added to that which arrived, and it is put on the same footing by the subtraction of all the charges which would have been incurred if it *had* arrived. As it did not arrive, there could be no duty or landing charges; and as the freight is usually paid on the delivery of the goods, the proprietor would not have sustained any loss by the payment of them. He obtains the net value of his goods from the contributory interests as general average, but the value which he receives is added to the value of the cargo in the apportionment; by which means he pays his share of his own loss or sacrifice.

Where a general average act occurs soon after leaving the port of departure, and the ship puts back, the invoice values are then taken, with the addition of all the shipping expenses, for they are part of the merchant's risk. They are in the price which the goods must realize at the market for which they are destined, and that is the value which is at risk; for if the goods were jettisoned, the same or equivalent charges would have to be incurred to ship merchandize of the like nature, which would be substituted for that sacrificed. The average is in that case usually adjusted before the vessel puts to sea again; and if the freight has been paid in advance, it is added to the charges on the cargo, for the merchant having paid it, regards it as one of the shipping expenses. It then forms part of *his* risk, and not that of the shipowner.

We have now seen that, by the process of bonification, any benefit that the ship may have derived from repairs which are paid for as general average, are added to the value for contribution, and that the freight and cargo jettisoned are likewise added to the arrived values, estimated for the same purpose; so that the three interests are as nearly as possible put upon one basis, each paying a proportion of the specific benefit derived.

One of the results of the meeting at York in 1864 was, that the delegates from foreign countries decided in favour of the mode adopted in England in valuing the property for contribution, with one slight alteration, namely, that the deduction to be made from the shipowner's freight and passage-money at risk should consist of two-fifths of such freight, in lieu of wages, port charges, and all other deductions.

From the foregoing remarks it will be manifest, that in order to arrive at an equitable distribution of the losses or expenses, the assistance of a third party must be obtained. In the first place, those upon whom the claim is made for compensation would scarcely be willing to abide by the decision of the person who made the demand, however large his experience or extensive his knowledge of those matters, although they might at the same time have no doubt of his integrity and honour; and in the next place, in order to obtain a proper settlement—especially in a case which involved any question of law or precedent—the person who acted in that capacity must be one thoroughly conversant with the principles and practices which govern general average and marine insurance, which knowledge few could acquire but those who devoted all their time and attention to the subject. The party who so acts is, in England, the colonies, and the United States, called an average-adjuster or stater; on the continent he is usually styled a *dispacheur*. His business is to state the claims and show the proper application of the losses and expenses which have been here considered. Another branch of his business is to state those claims which are made for loss or deterioration by sea damage on such marine policies as become liable by the terms on which they are issued; and this forms no inconsiderable portion of his duties.

This profession does not appear to have been of ancient date. Mr. Stevens is said to have been the first who followed it in London. The first edition of his *Essay on Average* is dated 1813. In it he mentions that in 1719 a particular average on *linens* was adjusted at Hamburgh by a "*dispacheur* of eminence," one Jurgen Greve; so that it would seem that the business was not sufficient to afford adequate remuneration when followed exclusively; and, indeed, such is the fact at the present day all over the continent. No doubt averages had been adjusted in London and other large commercial ports; but if the works published and the laws and ordinances promulgated may be taken as any criteria, the commerce of England did not at that time afford enough employment for the vocation to be recognized and ranked amongst those generally followed. In many parts of the United States of America and our colonies it is still associated with the business of an insurance broker. The first English work on insurance was published in 1755, and was followed at various intervals by ten other treatises on maritime law, laws of shipping, and marine insurance. The last of these bears the date 1802, and six years after Annesley wrote a book, in which (according to its title) "the mode of calculating averages is defined and illustrated by examples." From this it is clear that average claims were then beginning to be noticed, and that the mode of stating them was according to some systematic arrangement.

It is curious to observe the difference between the English and foreign average statements. Take, for example, a French general average statement. After reciting some of the facts of the case, three questions are proposed—(1) as to the nature of the loss or damage; (2) the sums applicable to each species of loss; (3) how ought they to be distributed? Following these are the "motifs sur la question." Then an elaborate answer to the first, followed by the actual statement; and the whole concluded by answering the last question, which is in effect a lengthened exposition of the apportionment of the general average. This is the substance of an adjustment at Havre. If the same set of papers had been placed in the hands of a London or Liverpool adjuster, the end would have been attained by a preamble of moderate length stating the facts which gave rise to the claim; this would be followed by the losses and disbursements, and, as briefly as possible, the apportionment according to the contributory values.

It is not intended by these observations to make any invidious comparison. The difference of treatment arises from two causes. The limited extent of the commercial relations of French, Italian, and other continental seaports, does not necessitate that expeditious mode of procedure without which, in Great Britain and the United States, business could not be effectively conducted. In spite of the concise method of settlement usually adopted in these countries, some average statements are of great length, owing to the complicated nature of the circumstances connected with them, and the number of parties interested in the cargo; so that if the repetition and documentary testimony, such as is customary on the continent, were to be introduced, many of them would be almost interminable. It is now the custom in England to print the long statements, as that plan is found to be more convenient in obtaining a speedy settlement; for where there are many consignees of the cargo, months may elapse before it is received by some of them; and the delay is especially inconvenient where the shipowner has a large sum to be refunded for his disbursements, or where money is due to the owners of the cargo for jettison, or for sale of part of the interest at an intermediate port. Where the statement is printed, each party has a copy, the expense of so doing having been previously charged as general average.

Another reason for this, and dependent in some measure upon the last, exists in the fact that although the *dispacheur* sometimes states the claims, his profession is quite distinct from that of an *average stater;* although it approximates more nearly than any other to the vocation followed and so styled in other countries.

It appears that in France (the greatest commercial country of Europe except Great Britain, and therefore a fair type of the rest), the *dispacheurs* are selected by the Insurance Companies of Paris, from the most talented of their directors; and that these gentlemen, like their compeers in London, can be numbered by units. They are, however, not resorted to in the first instance, as is the case elsewhere. There are certain rules prescribed by law which the merchant, ship-broker, or other person, applies according to the liability of ship, cargo, or other interest. Where a settlement cannot be thus obtained, recourse is had to the *dispacheur;* and when he has arranged the matter it is presented and concluded before the tribunal appointed for such purposes.

In France, before a person can become an advocate, notary, or ship-broker—of which callings the profession of a *dispacheur* largely partakes—there are certain qualifications necessary. Any one wishing to engage in either of these, or other similar branches of business, has to comply with two conditions. He must show that he is properly qualified by having received a liberal education; and in some cases it is advantageous, if not essential, that he should have taken the degree of bachelor of science. He is then permitted by the Government ("*sa Majesté*") to follow the vocation he may have selected.*

In England and the United States the requisite papers are placed at once in the hands of the average stater, who makes out the statement, which is immediately put in course of settlement. No stated rules are followed, like those just alluded to; and in no instance would the merchant or ship-broker put forward a claim of any importance without consulting those whose knowledge of maritime law, average, insurance, and nautical phraseology, enabled them to do justice to all parties concerned. It will have been noticed that general average *is* governed by certain laws, and that under many circumstances fixed rules are applicable; but they are not laid down as such, so that it would be easy for any one to apply them, for respecting the greater number it may be said that circumstances govern cases.

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* These particulars have been kindly communicated by a gentleman residing at St. Vaast-la-Hogue, near Cherbourg.

Where there has been no previous arrangement, a settlement is obtained at the port of discharge, according to the law of that port. This has regard to the respective owners of the ship and cargo; but the rule admits of modifications, according to the terms of the contract entered upon. The shipowner is usually the disburser and receiver in all matters connected with the general average.

A Budget of Paradoxes. By PROFESSOR DE MORGAN.

(Continued from page 54.)

No. XV. 1839—1840.

Old and new logic contrasted: being an attempt to elucidate, for ordinary comprehension, how Lord Bacon delivered the human mind from its 2,000 years' enslavement under Aristotle. By Justin Brenan. London, 1839, 12mo.

Logic, though the other exact science, has not had the sort of assailants who have clustered about mathematics. There is a sect which disputes the utility of logic, but there are no special points, like the quadrature of the circle, which excite dispute among those who admit other things. The old story about Aristotle having one logic to trammel us, and Bacon another to set us free—always laughed at by those who really knew either Aristotle or Bacon—now begins to be understood by a large section of the educated world. The author of this tract connects the old logic with the indecencies of the classical writers, and the new with moral purity: he appeals to women, who, "when they see plainly the demoralizing tendency of syllogistic logic, they will, no doubt, exert their powerful influence against it, and support the Baconian method." This is the only work against logic which I can introduce, but it is a rare one, I mean in contents. I quote the author's idea of a syllogism:—

> "The basis of this system is the syllogism. This is a form of couching the substance of your argument or investigation into one short line or sentence—then corroborating or supporting it in another, and drawing your conclusion or proof in a third."

On this definition he gives an example, as follows: "Every sin deserves death," the substance of the "argument or investigation." Then comes, "Every unlawful wish is a sin," which "corroborates or supports" the preceding; and, lastly, "therefore every unlawful wish deserves death," which is the "conclusion or proof." We learn, also, that "sometimes the first is called the premises (*sic*), and sometimes the first premiss": as also that "the first is sometimes called the proposition, or subject, or affirmative, and the next the predicate, and sometimes the middle term." To which is added, with a mark of exclamation at the end, "but, in analyzing the syllogism, there is a middle term, and a predicate too, in each of the lines!" It is clear that Aristotle never enslaved this mind.

Theory of Parallels. The proof of Euclid's axiom looked for in the properties of the equiangular spiral. By Lieut.-Col. T. Perronet Thompson. The same, 2nd edition, revised and corrected. The same, third edition, shortened, and freed from dependence on the theory of limits. The same, fourth edition, ditto, ditto. All London, 1840, 8vo.

To explain these editions it should be noted that General Thompson rapidly modified his notions, and republished his tracts accordingly.

Vestiges of the Natural History of Creation. London, 1840, 12mo.

This is the first edition of this celebrated work. Its form is a case of the theory: the book is an undeniable duodecimo, but the size of its paper gives it the look of not the smallest of octavos. Does not this illustrate the law of development, the gradation of families, the transference of species, and so on? If so, I claim the discovery of this esoteric testimony of the book to its own contents: I defy any one to point out the reviewer who has mentioned it. The work itself is described by its author as "the first attempt to connect the natural sciences into a history of creation." The attempt was commenced, and has been carried on, both with marked talent, and will be continued. Great advantage will result: at the worst we are but in the alchemy of some new chemistry, or the astrology of some new astronomy. Perhaps it would be as well not to be too sure on the matter until we have an antidote to possible consequences as exhibited under another theory, on which it is as reasonable to speculate as on that of the *Vestiges*. I met long ago with a splendid player on the guitar, who assured me, and was confirmed by his friends, that he *never practised*, except in thought, and did not possess an instrument: he kept his fingers acting in his mind, until they got their habits: and thus he learnt the most difficult novelties of execution. Now what if this should be a minor segment of a higher law? What if by constantly thinking of ourselves as descended from primæval monkeys, we should—if this be true—actually *get our tails again?* What if the first man who was detected with such an appendage should be obliged to confess himself the author of the *Vestiges*—a personage yet unknown—who would naturally get the start of his species by having had the earliest habit of thinking on the matter? I confess I never hear a man of note talk fluently about it without a curious glance at his proportions, to see whether there may be ground to conjecture that he may have more of "mortal coil" than others, in anaxyridical concealment. I do not feel sure that even paternal love for his theory would induce him, in the case I am supposing, to exhibit himself at the British Association,

> "With a hole behind which his tail peeped through."

The first sentence of this book (1840) is a cast of the log, which shows our rate of progress. "It is familiar knowledge that the earth which we inhabit is a globe of somewhat less than 8,000 miles in diameter, being one of a series of eleven which revolve at different distances around the sun." The *eleven!* Not to mention the Iscariot which Leverrier and Adams calculated into existence, there is more than a septuagint of *new* planetoids.

The Constitution and rules of the ancient and universal "Benefit Society" established by Jesus Christ, exhibited, and its advantages and claims maintained, against all modern and merely human Institutions of the kind: A letter very respectfully addressed to the Rev. James Everett, and occasioned by certain remarks made by him, in a speech to the Members of the "Wesleyan Centenary Institute" Benefit Society. Dated York, Dec. 7, 1840. By Thomas Smith. 12mo. (pp. 8.)

The Wesleyan minister addressed had advocated provision against old age, &c.: the writer declares all *private* provision unchristian. After decent maintenance and relief of family claims of indigence, he holds that all the rest is to go to the "Benefit Society" of which he draws up the rules, in technical form, with chapters of "Officers," "Contributors," &c., from the Acts of the Apostles, &c., and some of the early Fathers. He holds that a Christian may not "make a *private* provision against the contingencies of the future": and that the great "Benefit Society" is the divinely-ordained recipient of all the surplus of his income; capital, beyond what is necessary for business, he is to have none. A real good speculator shuts his eyes by instinct, when opening them would not serve the purpose: he has the vizor of the Irish fairy tale, which fell of itself over the eyes of the wearer the moment he turned them upon the enchanted light which would have destroyed him if he had caught sight of it. "Whiles it remained, was it not thine own? and after it was sold, was it [the purchase-money] not in thine own power?" would have been awkward to quote, and accordingly nothing is stated except the well-known result, which is rule 3, cap. 5, "Prevention of Abuses." By putting his principles together, the author can be made, logically, to mean that the successors of the apostles should put to death all contributors who are detected in not paying their full premiums.

I have known one or two cases in which policy-holders have surrendered their policies through having arrived at a conviction that direct provision is unlawful. So far as I could make it out, these parties did not think it unlawful to lay by out of income, except when this was done in a manner which involved calculation of death-chances. It is singular that they did not see that the entrance of chance of death was the entrance of the very principle of the Benefit Society described in the Acts of the Apostles. The family of the one who died young received more in proportion to *premiums* paid than the family of the one who died old. Every one who understands life assurance sees that—*bonus* apart—the difference between an Assurance Office and a Savings Bank consists in the adoption, *pro tanto*, of the principle of community of goods. In the original constitution of the oldest Assurance Office, the *Amicable Society*, the plan with which they started was nothing but this: persons of all ages under forty-five paid one common premium, and the proceeds were divided among the representatives of those who died within the year.

No. XVI. 1842—1845.

1842. Provisional Prospectus of the Double Acting Rotary Engine Company. Also Mechanic's Magazine, March 26, 1842.

Perpetual motion by a drum with one vertical half in mercury, the other in a vacuum: the drum, I suppose, working round for ever to find an easy position. Steam to be superseded: steam and electricity convulsions of nature never intended by Providence for the use of man. The price of the present engines, as old iron, will buy new engines that will work without fuel and at no expense. Guaranteed by the Count de Predaval, the discoverer. I was to have been a Director, but my name got no further than ink, and not so far as official notification of the honour, partly owing to my having communicated to the *Mechanic's Magazine* information privately given to me, which gave premature publicity, and knocked up the plan.

An exposition of the nature, force, action, and other properties of gravitation on the planets. London, 1842, 12mo.

An investigation of the principles of the rules for determining the measures of the areas and circumferences of circular plane surfaces.... London, 1844, 8vo.

These are anonymous; but the author (whom I believe to be Mr. Denison, presently noted) is described as author of a new system of mathematics, and also of mechanics. He had need have both, for he shows that the line which has a square equal to a given circle, has a cube equal to the sphere on the same diameter: that is, in old mathematics, the diameter is to the circumference as 9 to 16! Again, admitting that the velocities of planets in circular orbits are inversely as the square roots of their distances, that is, admitting Kepler's law, he manages to prove that gravitation is inversely as the square *root* of the distance: and suspects magnetism of doing the difference between this and Newton's law. Magnetism and electricity are in physics the member of parliament and the cabman; at every man's bidding, as Henry Warburton said.

The above is an outrageous quadrature. In the preceding year, 1841, was published what I supposed at first to be a Maori quadrature, by Maccook. But I get it from a cutting out of some French periodical, and I incline to think that it must be by a Mr. M'Cook. He makes π to be $2+2\sqrt{(8\sqrt{2}-11)}$.

Refutation of a pamphlet written by the Rev. John Mackey, R.C.P., entitled "A method of making a cube double of a cube, founded on the principles of elementary geometry," wherein his principles are proved erroneous, and the required solution not yet obtained. By Robert Murphy. Mallow, 1824, 12mo.

This refutation was the production of an Irish boy of eighteen years old, self-educated in mathematics, the son of a shoemaker at Mallow. He died in 1843, leaving a name which is well known among mathematicians. His works on the theory of equations and on electricity, and his papers in the *Cambridge Transactions*, are all of high genius. The only account of him which I know of is that which I wrote for the Supplement of the *Penny Cyclopædia*. He was thrown by his talents into a good income at Cambridge, with no social training except penury, and very little intellectual training except mathematics. He fell into dissipation, and his scientific career was almost arrested: but he had great good in him, to my knowledge. A sentence in a letter from the late Dean Peacock to me—giving some advice about the means of serving Murphy—sets out the whole case: "Murphy is a man whose *special* education is in advance of his *general*; and such men are almost always difficult subjects to manage." This article having been omitted in its proper place, I put it at 1843, the date of Murphy's death.

The invisible universe disclosed: or the real plan and government of the Universe. By Henry Coleman Johnson, Esq. London, 1843, 8vo.

The book opens abruptly with—

"First demonstration. Concerning the centre: showing that, because the centre is an innermost point at an equal distance between two extreme points of a right line, and from every two relative and opposite intermediate points, it is composed of the two extreme internal points of each half of the line; each extreme internal point attracting towards itself all parts of that half to which it belongs...."

Of course the circle is squared: and the circumference is $3\frac{1}{21}$ diameters.

Combination of the zodiacal and cometical systems. Printed for the London Society, Exeter Hall. Price Sixpence. (n. d. 1843).

What this London Society was, or the "combination," did not appear. There was a remarkable comet in 1843, the tail of which was at first confounded with what is called the *zodiacal light*. This nicely-printed little tract, evidently got up with less care for expense than is usual in such works, brings together all the announcements of the astronomers, and adds a short head and tail piece, which I shall quote entire. As the announcements are very ordinary astronomy, the reader will be able to detect, if detection be possible, what is the meaning and force of the "Combination of the zodiacal and cometical systems":—

"*Premonition.* It has pleased the AUTHOR OF CREATION, to cause (to His *human and reasoning* Creatures of this generation, by a '*combined*' appearance in His *Zodiacal* and *Cometical* systems,) a '*warning Crisis*,' of universal concernment to this our GLOBE. It is this '*Crisis*' that has so generally 'ROUSED' at this moment the '*nations throughout the Earth*' that no equal interest has ever before been excited by MAN; unless, it be in that caused by the 'PAGAN-TEMPLE IN ROME,' which is recorded by the elder Pliny, '*Nat. Hist.*' i. 23. iii. 3. HARDOUIN."

—After the accounts given by the unperceiving astronomers, comes what follows:—

"Such has been (*hitherto*) the only object discerned by the '*Wise of this World*,' in this *two-fold union* of the '*Zodiacal* and *Cometical*' systems: yet it is nevertheless, a most '*Thrilling Warning*' to *all* the inhabitants of this precarious and transitory Earth. We have no authorized intimation, or reasonable prospective contemplation, of '*current time*' beyond a year 1860, of the present century; or rather, except '*the interval which may now remain from the present year* 1843, *to a year* 1860 (ἡμερας ΕΞΗΚΟΝΤΑ—'*threescore or sixty days*'—'*I have appointed each* "Day" *for a* "Year,"' *Ezek.* iv. 6): and we know, from our '*common experience*' how speedily such a measure of time will pass away.

"No words can be '*more explicit*,' than these of our blessed Lord: viz. 'This Gospel *of the Kingdom shall be preached in* all the Earth, *for a Witness to* all Nations; and then, *shall the* End come.' The '*next* 18 *years*,' must therefore supply the interval of the '*special Episcopal forerunners*.'"

(Matt. xxiv. 14.)

"See, the 'Jewish Intelligencer' of the present month (*April*), p. 153, for the '*Debates in Parliament*,' respecting the Bishop of Jerusalem, *viz.*, Dr. Bowring, Mr. Hume, Sir R. Inglis, Sir R. Peel, Viscount Palmerston."

I have quoted this at length, to show the awful threats which were published at a time of some little excitement about the phenomenon, under the name of the *London Society*. The assumption of a corporate appearance is a very unfair trick: and there are junctures at which harm might be done by it.

Wealth the name and number of the Beast, 666, in the book of Revelation. [By John Taylor.] London, 1844, 8vo.

Whether Junius or the Beast be the more difficult to identify, must be referred to Mr. Taylor, the only person who has attempted both. His cogent argument on the political secret is not unworthily matched in his treatment of the theological riddle. He sees the solution in εὐπορία, which occurs in the Acts of the Apostles as the word for wealth in one of its most disgusting forms, and makes 666 in the most straightforward way. This explanation has as good a chance as any other. The work contains a general attempt at explanation of the Apocalypse, and some history of opinion on the subject. It has not the prolixity which is so common a fault of apocalyptic commentators.

A practical Treatise on Eclipses....with remarks on the anomalies of the present Theory of the Tides, by T. Kerigan, F.R.S. 1844, 8vo.

Containing also a refutation of the theory of the tides, and afterwards increased by a supplement, "Additional facts and arguments against the theory of the tides," in answer to a short notice in the *Athenæum* journal. Mr. Kerigan was a lieutenant in the Navy: he obtained admission to the Royal Society just before the publication of his book.

A new theory of Gravitation. By Joseph Denison, Esq. London, 1844, 12mo.
Commentaries on the Principia. By the author of "A new theory of Gravitation." London, 1846. 8vo.

Honour to the speculator who can be put in his proper place by one sentence, be that place where it may.

> "But we have shown that the velocities are inversely as the square roots of the mean distances from the sun; wherefore, by equality of ratios, the forces of the sun's gravitation upon them are also inversely as the square roots of their distances from the sun."

The discovery of a general resolution of all superior finite equations, of every numerical both algebraick and transcendent form. By A. P. Vogel, mathematician at Leipzick. Leipzick and London, 1845, 8vo.

This work is written in the English of a German who has not mastered the idiom: but it is always intelligible. It professes to solve equations of every degree "in a more extent sense, and till to every degree of exactness." The general solution of equations of *all* degrees is a vexed question, which cannot have the mysterious interest of the circle problem, and is of a comparatively modern date. Mr. Vogel announces a forthcoming treatise in which are resolved the "last impossibilities of pure mathematics."

Elective Polarity the Universal Agent. By Frances Barbara Burton, authoress of "Astronomy familiarized," "Physical Astronomy," &c. London, 1845, 8vo.

The title gives a notion of the theory. The first sentence states, that 12,500 years ago *a* Lyræ was the pole-star, and attributes the immense magnitude of the now fossil animals to a star of such "polaric intensity as Vega pouring its magnetic streams through our planet." Miss Burton was a lady of property, of estimable character, and of very respectable acquirements, especially in Hebrew; she was eccentric in all things.

No. XVII. 1846—1847.

A new theory of the tides: in which the errors of the usual theory are demonstrated; and proof shewn that the full moon is not the cause of a concomitant spring tide, but actually the cause of the neaps ... By Commr. Debenham, R.N. London, 1846, 8vo.

The author replied to a criticism in the *Athenæum*, and I remember how, in a very few words, he showed that he had read nothing on the subject. The reviewer spoke of the forces of the planets (*i.e.*, the Sun and Moon) on the Ocean, on which the author remarks, "But N.B. the Sun is no planet, Mr. Critic." Had he read any of the actual investigations on the usual theory, he would have known that to this day the sun and moon continue to be called *planets*—though the phrase is disappearing—in speaking of the tides; the sense, of course, being the old one, wandering bodies.

A large class of the paradoxers, when they meet with something which taken in their sense is absurd, do not take the trouble to find out the intended meaning, but walk off with the words laden with their own first construction. Such men are hardly fit to walk the streets without an interpreter. I was startled for a moment, at the time when a recent happy—and more recently happier—marriage occupied the public thoughts, by seeing in a haberdasher's window, in staring large letters, an unpunctuated sentence which read itself to me as "Princess Alexandra! collar and cuff!" It immediately occurred to me that had I been any one of some scores out of my paradoxers, I should, no doubt, have proceeded to raise the mob against the unscrupulous person who dared to hint to a young bride such maleficent—or at least immellificent—conduct towards her new lord. But, as it was, certain material contexts in the shop window suggested a less savage explanation. A paradoxer should not stop at reading the advertisements of Newton or Laplace: he should learn to look at the stock of goods.

Two systems of astronomy: first, the Newtonian system, showing the rise and progress thereof, with a short historical account; the general theory with a variety of remarks thereon: second, the system in accordance with the Holy Scriptures, showing the rise and progress from Enoch, the seventh from Adam, the prophets, Moses, and others, in the first Testament; our Lord Jesus Christ, and his apostles, in the new or second Testament; Reeve and Muggleton, in the third and last Testament; with a variety of remarks thereon. By Isaac Frost. London, 1846, 4to.

A very handsomely printed volume, with beautiful plates. Many readers who have heard of Muggletonians have never had any distinct idea of Lodowick Muggleton, the inspired tailor (1608—1698), who about 1650 received his commission from heaven, wrote a Testament, founded a sect, and descended to posterity. Of Reeve less is usually said; according to Mr. Frost, he and Muggleton are the two "witnesses." I shall content myself with one specimen of Mr. Frost's science.

"I was once invited to hear read over 'Guthrie on Astronomy,' and when the reading was concluded I was asked my opinion thereon; when I said, 'Doctor, it appears to me that Sir I. Newton has only given two proofs in support of his theory of the earth revolving round the sun: all the rest is assertion without any proofs.'—'What are they?' inquired the Doctor.—'Well, I said, 'they are, first, the power of attraction to keep the earth to the sun; the second is the power of repulsion, by virtue of the centrifugal motion of the earth: all the rest appears to me assertion without proof.' The Doctor considered a short time, and then said, 'It certainly did appear so.' I said, 'Sir Isaac has certainly obtained the credit of completing the system, but really he has only half done his work.'—'How is that,' inquired my friend the Doctor. My reply was this: 'You will observe his system shows the earth traverses round the sun on an inclined plane; the consequence is, there are four powers required to make his system complete:

1st. The power of *attraction.*
2ndly. The power of *repulsion.*
3rdly. The power of *ascending* the inclined plane.
4thly. The power of *descending* the inclined plane.

You will thus easily see the *four* powers required, and Newton has only accounted for *two;* the work is therefore only half done.' Upon due reflexion the Doctor said, 'It certainly was necessary to have these *four* points cleared up before the system could be said to be complete.'"

I have no doubt that Mr. Frost, and many others on my list, have really encountered doctors who could be puzzled by such stuff as this, or nearly as bad, among the votaries of existing systems, and have been encouraged thereby to print their objections. But justice requires me to say that from the words "power of repulsion by virtue of the centrifugal motion of the earth," Mr. Frost may be suspected of having something more like a notion of the much-mistaken term "centrifugal force" than many paradoxers of greater fame. The Muggletonian sect is not altogether friendless: over and above this handsome volume, the works of Reeve and Muggleton were printed in 1832, in three quarto volumes. See *Notes and Queries,* 1st Series, v. 80; 3rd Series, iii. 303.

Astronomical Aphorisms, or Theory of Nature; founded on the immutable basis of Meteoric Action. By P. Murphy, Esq. London, 1847, 12mo.

This is by the framer of the Weather Almanac, who appeals to that work as corroborative of his theory of planetary temperature, years after all the world knew by experience that this meteorological theory was just as good as the others.

The conspiracy of the Bullionists as it affects the present system of the money laws. By Caleb Quotem. Birmingham, 1847, 8vo. (pp. 16).

This pamphlet is one of a class of which I know very little, in which the effects of the laws relating to this or that political bone of contention are imputed to deliberate conspiracy of one class to rob another of what the one knew ought to belong to the other. The success of such writers in believing what they have a bias to believe, would, if they knew themselves, make them think it equally likely that the inculpated classes might really believe what it is *their* interest to believe. The idea of a *guilty* understanding existing among fundholders, or landholders, or any holders, all the country over, and never detected except by bouncing pamphleteers, is a theory which should have been left for Cobbett to propose, and for Apella to believe.

The Reasoner. No. 45. Edited by G. J. Holyoake. Price 2*d.* Is there sufficient proof of the existence of God? 8vo, 1847.

This acorn of the holy oak was forwarded to me with a manuscript note, signed by the editor, on the part of the London Society of Theological Utilitarians," who say "they trust you may be induced to give this momentous subject your consideration." The supposition that a middle-aged person, known as a student of thought on more subjects than one, had that particular subject yet to begin, is a specimen of what I will call the *assumption-trick* of controversy, a habit which pervades all sides of all subjects. The tract is a proof of the good policy of letting opinions find their level, without any assistance from the Court of Queen's Bench. Twenty years earlier the thesis would have been positive, "There is sufficient proof of the non-existence of God," and bitter in its tone. As it stands, we have a moderate and respectful treatment —wrong only in making the opponent argue absurdly, as usually happens when one side invents the other—of a question in which a great many Christians have agreed with the atheist: that question being—Can the existence of God be proved independently of revelation? Many very religious persons answer this question in the negative, as well as Mr. Holyoake. And, this point being settled, all who agree in the negative separate into those who can endure scepticism, and those who cannot: the second class find their way to Christianity. This very number of *The Reasoner* announces the secession of one of its correspondents, and his adoption of the Christian faith. This would not have happened twenty years before: nor, had it happened, would it have been respectfully announced.

The booby notion that the *non*-existence of God can be *proved*, has died out under the light of discussion: had the only lights allowed shone from the pulpit and the prison, so great a step would never have been made. The question now is as above. The dictum that Christianity is "part and parcel of the law of the land" is also abrogated: at the same time, and the coincidence is not an accident, it is becoming somewhat nearer the truth that the law of the land is part and parcel of Christianity. It must also be noticed that *Christianity* was part and parcel of the articles of *war*; and so was *duelling*. Any officer speaking against religion was to be cashiered; and any officer receiving an affront without, in the last resort, attempting to kill his opponent, was also to be cashiered. Though somewhat of a book-hunter, I have never been able to ascertain the date of the collected remonstrances of the prelates in the House of Lords against this overt inculcation of murder, under the soft name of *satisfaction*: it is neither in Watt, nor in Lowndes, nor in any edition of Brunet; and there is no copy in the catalogue of the British Museum. Was the collected edition really published?

With the general run of the philosophical atheists of the last century the notion of a God was an hypothesis. There was left an admitted possibility that the vague somewhat which went by more names than one, might be personal, intelligent, and superintendent. In the works of Laplace, who is sometimes called an atheist from his writings, there is nothing from which such an inference can be drawn: unless indeed a reverend Fellow of the Royal Society may be held to be the fool who said in his heart, &c. &c., if his contributions to the *Philosophical Transactions* go no higher than *nature*. The following anecdote is well known in Paris, but has never been printed entire. Laplace once went in form to present some edition of his *Systême du Monde* to the First Consul, or Emperor. Napoleon, whom some wags had told that this book contained no mention of the name of God, and who was fond of putting embarrassing questions, received it with—"M. Laplace, they tell me you have written this large book on the system of the universe, and have never even mentioned its Creator." Laplace, who though the most supple of politicians, was as stiff as a martyr on every point of his philosophy or religion (*ex. gr.*, even under Charles the Tenth he never concealed his dislike of the priests), drew himself up, and answered bluntly, "Je n'avais pas besoin de cette hypothèse-là." Napoleon, greatly amused, told this reply to Lagrange, who exclaimed, "Ah! c'est une belle hypothèse; ça explique beaucoup de choses."

An examination of the Astronomical doctrine of the Moon's rotation. By G. L. Edinburgh, 1847, 8vo.

A systematic attack of the character afterwards made with less skill and more notice by Mr. Jellinger Symons.

Miracles *versus* Nature: being an application of certain propositions in the theory of chances to the Christian miracles. By Protimalethes. Cambridge, 1847, 8vo.

The theory, as may be supposed, is carried further than most students of the subject would hold defensible.

An astronomical Lecture. By the Rev. R. Wilson. Greenock, 1847, 12mo.

Against the moon's rotation on her axis.

[Handed about in the streets in 1847: I quote the whole:] Important discovery in astronomy, communicated to the Astronomer Royal, December 21st, 1846. That the Sun revolve round the Planets in 25748$\frac{2}{8}$ years, in consequence of the combined attraction of the planets and their satellites, and that the Earth revolve round the Moon in 18 years and 228 days. D. T. GLAZIER [altered with a pen into GLAZION.] Price one penny.

1847. In the *United Service Magazine* for September, 1847, Mrs. Borron, of Shrewsbury, published some remarks tending to impeach the fact that Neptune, the planet found by Galle, really was the planet which Leverrier and Adams had a right to claim. This was followed (September 14) by two pages, separately circulated, of *Further Observations upon the Planets Neptune and Uranus, with a Theory of Perturbations;* and (October 19, 1848) by three pages of *A Review of M. Leverrier's Exposition.* Several persons, when the remarkable discovery was made, contended that the planet actually discovered was an intruder; and the future histories of the discovery must contain some account of this little after-piece. Tim Linkinwater's theory that there is no place like London for coincidences, would have been utterly overthrown in favour of what they used to call the celestial spaces, if there had been a planet which by chance was put near the place assigned to Neptune at the time when the discovery was made.

(*To be continued.*)

On the Value of Annuities payable Half-yearly, Quarterly, &c. By T. B. SPRAGUE, ESQ., M.A., *Actuary of the Equity and Law Life Assurance Society.*

THIS subject has been so fully treated by many writers, that it may be thought little or nothing new can be said in reference to it. It may, however, be useful to bring together and to compare the results obtained by the standard writers on the theory of life contingencies, to examine how far those results agree with each other, and when they differ, to ascertain the causes of the difference.

When an annuity of £1 is payable m times in the course of the year, the common rule for finding its value is to add $\frac{m-1}{2m}$ to the value of the yearly annuity. This may be easily arrived at in the following way. The age of the life on which the annuity depends being k, the value of an annuity of £1, of which the first payment is made at once, is $1+a_k$; and the value of the same annuity when the first payment is made at the end of a year, is a_k; therefore the value of the same annuity when the first payment is made at the end of the $\frac{1}{m}$th part of a year, will be $a_k+\frac{m-1}{m}$ approximately. If there are m such annuities of which the first payments are made at the end of the $\frac{1}{m}$th, $\frac{2}{m}$th, $\frac{3}{m}$th .... portions of a year respectively, the sum of their values will be approximately

$$\left(a_k+\frac{m-1}{m}\right)+\left(a_k+\frac{m-2}{m}\right)+\left(a_k+\frac{m-3}{m}\right)+\ldots+a_k$$
$$=ma_k+\frac{(m-1)m}{2m}.$$

Whence, dividing by m, the value of an annuity of £1 payable by instalments of $\frac{1}{m}$ at equal intervals throughout each year, is $a_k+\frac{m-1}{2m}$ approximately.

This demonstration does not enable us to judge of the closeness of the approximation; but this is equally true of Mr. Milne's demonstration, which is very much more lengthy without being more satisfactory. Mr. Milne begins by finding the value of an annuity according to De Moivre's hypothesis—first, when payable annually; secondly, when payable by m equal instalments; and proves that the latter value is greater then the former by $\frac{m-1}{2m}$ approximately. He then establishes this "Lemma":—"If q and q' be any two quantities, known or unknown, that are nearly equal, and p, p', their respective near values, both determined by the same method of approximation, then will $p-p'$ be much nearer to $q-q'$ than either of the proximate values is to the true." From this it follows that the true value of the annuity when payable m times a year exceeds the true value of the annuity payable yearly by $\frac{m-1}{2m}$ nearly.

In reference to this rule, Professor De Morgan remarks (*Companion to the Almanac* for 1842, p. 9):—"When an annuity is to be paid, not yearly, but half-yearly or quarterly, &c., *interest being convertible into principal only once a year,* the common correction of A, the present value of a yearly annuity, is to write $A+\frac{n-1}{2n}$, instead of A, n being the number of times which the annuity is payable yearly. This correction is strictly true when we speak of a perpetual annuity, on the supposition that by interest being convertible once a year, we mean that money may be invested at any one time of the year, so that the grantor of the annuity must be held to lose simple interest for the rest of the year on each of the prepaid portions." The limitation expressed by the words which I have placed in italics, is, however, unnecessary; for the correction applies equally whether the interest be convertible only once a year or n times a year.

Mr. Baily has not considered the question generally, but has given the exact formulæ for the values of annuities payable half-yearly and quarterly. His formula for the value of an annuity payable half-yearly, which is based upon the usual supposition that the deaths are distributed uniformly over each year, is

$$\frac{\sqrt{v}}{4}\{2(1+i)^{\frac{1}{2}}+2+i\}a_k+\frac{\sqrt{v}}{4},$$

which is equal to

$$\frac{a_k}{4}\left\{2+\sqrt{1+i}+\frac{1}{\sqrt{1+i}}\right\}+\frac{1}{4\sqrt{1+i}}$$

$$=\frac{a_k}{4}\left\{2+1+\frac{i}{2}-\frac{i^2}{8}+\frac{i^3}{16}-\ \ldots\ldots\right.$$

$$\left.+1-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ \ldots\ldots\right\}$$

$$+\frac{1}{4}\left\{1-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ \ldots\ldots\right\}$$

$$=\frac{a_k}{4}\left\{4+\frac{i^2}{4}-\frac{i^3}{4}+\ \ldots\ldots\right\}+\frac{1}{4}-\frac{i}{8}+\frac{3}{32}i^3-\ \ldots\ldots$$

$$=a_k+\frac{1}{4}, \text{ approximately,}$$

since i is usually a small fraction.

The above expansions enable us to estimate the closeness of the approximation given by the formula $a_k + \frac{1}{4}$. The errors caused by taking the first terms only of the series

$$4 + \frac{i^2}{4} - \frac{i^3}{4} + \ldots .$$

and

$$\frac{1}{4} - \frac{i}{8} + \frac{3}{32}i^2 - \ldots .$$

will be nearly equal to the first of the neglected terms in each case, *i.e.*, to $\frac{i^2}{4}$ in the former and $-\frac{i}{8}$ in the latter series. Assuming the rate of interest to be 3 per cent., $i = \cdot 03$, and $\frac{i^2}{4} = \cdot 000225$; and since a_k in practice is always less than 25, the error caused by neglecting the terms $\frac{i^2}{4} - \frac{i^3}{4} + \ldots$ will be less than ·0014, or can only amount to a unit in the third decimal place. Again, $\frac{i}{8} = \cdot 00375$, so that the error caused by neglecting the terms $-\frac{i}{8} + \frac{3}{32}i^2 - \ldots$ is of more importance; and if we require the third decimal place to be correct, we must take $a_k + \frac{1}{4} - \frac{i}{8}$ for the value of the annuity payable half-yearly.

Baily's formula for the value of an annuity payable quarterly is

$$a_k \times \frac{v^{\frac{3}{4}}}{16}\left\{4(1+i)^{\frac{3}{4}} + (4+i)(1+i)^{\frac{1}{2}} + 2(2+i)(1+i)^{\frac{1}{4}} + 4 + 3i\right\}$$

$$+ \frac{v^{\frac{3}{4}}}{16}\left\{3(1+i)^{\frac{1}{2}} + 2(1+i)^{\frac{1}{4}} + 1\right\},$$

which may be shown to be equal to

$$a_k\left\{1 + \frac{5i^2}{64} \ldots .\right\} + \frac{3}{8} - \frac{5}{32}i + \frac{15}{128}i^2 \ldots .$$

Here, again, the ordinary approximation, $a_k + \frac{3}{8}$, is true to the second, but not to the third decimal place.

In proving the above formulæ, Baily assumes that if i be the yearly rate of interest, then the value of a sum due in six months' time is $(1+i)^{-\frac{1}{2}}$; and of a sum due in three months, $(1+i)^{-\frac{1}{4}}$. The propriety of this assumption is called in question by Milne, and considerable difference of opinion appears to exist on the point. Without now discussing the question, it will suffice to say that my own opinion is decidedly in favour of Baily's method. Some years ago there was an animated correspondence on the subject in the *Assurance Magazine* (see Vols iii. and iv.). The argument in favour of Baily's method appears to me to have been most forcibly stated in the letter of Mr. Orchard, vol iv., p. 61, and I entirely endorse the reasoning of that letter.

The problem of finding the exact value of an annuity payable m times in each year, has been considered by Mr. Griffith Davies. He states the problem thus, p. 328 of his book:—"To find the present value of £1 annuity, payable by m equal instalments in each year, during the existence of a single life, A; supposing the interest convertible into principal at the like intervals, and the annuity to cease with the last instalment becoming due prior to the extinction of the life proposed." His solution is,

$$A^{(m)}=(P+Q)\mathbb{A}+P,$$

where $\mathbb{A}$ is the value of the £1 annuity payable yearly, calculated at that rate of interest which makes the amount of £1 in one year equal to $\left(1+\frac{i}{m}\right)^m$; also $P=\Sigma_x\frac{(m-x)m^{x-2}}{(m+i)^x}$, and $Q=\left(1+\frac{i}{m}\right)^m\Sigma_x\frac{xm^{x-2}}{(m+1)^x}$.

When we examine closely the reasoning by which this solution is obtained, we find that the suppositions made are virtually the same as those made by Baily; and accordingly the above solution must admit of being transformed into the very dissimilar looking ones given by Baily for the particular cases of half-yearly and quarterly payments.

Mr. Davies's reasoning is as follows:—The present age being k, assume, as usual, that the deaths in each year are distributed uniformly over the year, and that each year is divided into m equal portions; then the number of persons surviving the xth portion of the tth year will be

$$l_{k+t-1}-\frac{x}{m}(l_{k+t-1}-l_{k+t})=\frac{m-x}{m}l_{k+t-1}+\frac{x}{m}l_{k+t};$$

and the probability that a person of the age k will survive the xth portion of the tth year will be

$$\frac{m-x}{m}p_{k,t-1}+\frac{x}{m}p_{k,t}.$$

Also supposing interest convertible into principal m times in a year, the value of £1 due in a year's time is $\frac{1}{\left(1+\frac{i}{m}\right)^m}$, $=u$, suppose.

Then the value of the payment $\frac{1}{m}$ to be made at the end of the xth portion of the tth year will be

$$\frac{1}{m}\times\frac{u^{t-1}}{\left(1+\frac{i}{m}\right)^x}\times\left(\frac{m-x}{m}p_{k+t-1}+\frac{x}{m}p_{k,t}\right)$$

$$=\frac{(m-x)m^{x-2}}{(m+i)^x}\times u^{t-1}p_{k+t-1}+\frac{xm^{x-2}}{u(m+i)^x}\times u^t p_{k+t}$$

Therefore the value of all the payments to be made in the tth year will be

$$\Sigma_x\left\{\frac{(m-x)m^{x-2}}{(m+i)^x}\right\}u^{t-1}p_{k+t-1}+\Sigma_x\left\{\frac{xm^{x-2}}{u(m+i)^x}\right\}u^t p_{k+t},$$

x in the summation having the values 1, 2, 3 . . . m.

Put
$$\Sigma_x\left\{\frac{(m-x)m^{x-2}}{(m+i)^x}\right\}=\text{P},$$

$$\Sigma_x\left\{\frac{xm^{x-2}}{u(m+i)^x}\right\}=\text{Q}.$$

Then the value of all the payments falling due in the tth year will be

$$\text{P}.u^{t-1}p_{k+t-1}+\text{Q}u^t p_{k+t}.$$

Now giving t the values 1, 2, 3 . . . , and adding, we get the total value of the annuity

$$\text{P}\Sigma_t(u^{t-1}p_{k+t-1})+\text{Q}\Sigma^t(u^t p_{k+t}).$$

But
$$\Sigma_t(u^{t-1}p_{k+t-1})=1+\mathbb{A},$$

and
$$\Sigma_t(u^t p_{k+t})=\mathbb{A},$$

where $\mathbb{A}$ is the value of an annuity of £1 payable yearly, calculated at the rate of interest which makes the value of £1 in a year equal to $\left(1+\frac{i}{m}\right)^m$. Hence, finally, the value of the annuity required is $\text{A}^{(m)}=\text{P}(1+\mathbb{A})+\text{Q}\mathbb{A}$, or $(\text{P}+\text{Q})\mathbb{A}+\text{P}$.

The solution is not carried beyond this point by Mr. Davies, but he deduces the values of the annuity when paid half-yearly and quarterly, viz.:—

$$\text{A}^{(2)}=\left(\frac{1}{4+2i}\right)(1+\mathbb{A})+\frac{6+i}{8}\mathbb{A},$$

and
$$\text{A}^{(4)}=\frac{96+32i+3i^2}{4(4+i)^3}(1+\mathbb{A})+\frac{640+160i+20i^2+i^3}{1024}\mathbb{A}.$$

It is easy, however, to perform the summation indicated in the foregoing expressions for P and Q. Thus—

$$P=\Sigma_x\frac{(m-x)m^{x-2}}{(m+i)^x}$$

$$=\frac{1}{m}\Sigma_x\left(\frac{m}{m+i}\right)^x-\frac{1}{m^2}\Sigma_x x\left(\frac{m}{m+i}\right)^x$$

$$=\frac{1}{m}\Sigma_x c^x-\frac{1}{m^2}\Sigma_x xc^x,\text{ putting } c=\frac{m}{m+i}.$$

Now,
$$\Sigma_x c^x=c+c^2+\ldots\ldots+c^m$$

$$=\frac{c(1-c^m)}{1-c}$$

$$=\frac{m}{m+i}\cdot\frac{1-\left(\frac{m}{m+i}\right)^m}{1-\frac{m}{m+i}}$$

$$=\frac{m}{i}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}=\frac{m}{i}(1-u);$$

and
$$\Sigma_x xc^x=c+2c^2+3c^3+\ldots\ldots+mc^m$$

$$=\frac{c(1-c^m)}{(1-c)^2}-\frac{mc^{m+1}}{1-c}$$

$$=\frac{c}{(1-c)^2}\{1-(1+m)c^m+mc^{m+1}\}$$

$$=\frac{\frac{m}{m+i}}{\left(1-\frac{m}{m+i}\right)^2}\left\{1-u\left(1+m-\frac{m^2}{m+i}\right)\right\}$$

$$=\frac{m(m+i)}{i^2}\left\{1-u\left(1+\frac{mi}{m+i}\right)\right\}.$$

Hence
$$P=\frac{1}{m}\Sigma_x c^x-\frac{1}{m^2}\Sigma_x xc^x$$

$$=\frac{1}{i}(1-u)-\frac{m+i}{mi^2}\left\{1-u\left(1+\frac{mi}{m+i}\right)\right\}$$

$$=\frac{1}{i}\left(1-\frac{m+i}{mi}\right)-\frac{u}{i}\left(1-\frac{m+i}{mi}-1\right)$$

$$=\frac{1}{i}\left(1-\frac{m+i}{mi}\right)+\frac{u(m+i)}{mi^2}$$

$$=\frac{1}{i}-\frac{m+i}{mi^2}(1-u),$$

and
$$Q=\Sigma_x\frac{xm^{x-2}}{u(m+i)^x}=\frac{1}{m^2u}\Sigma_x\frac{xm^x}{(m+i)^x}$$

$$=\frac{m+i}{mui^2}-\frac{m+i}{mi^2}\left(1+\frac{mi}{m+i}\right)$$

$$=\frac{m+i}{mui^2}-\frac{1}{i}\left(\frac{m+i}{mi}+1\right);$$

and
$$\mathrm{P}+\mathrm{Q}=\frac{m+i}{mui^2}-\frac{2(m+i)}{mi^2}+\frac{u(m+i)}{mi^2}$$
$$=\frac{m+i}{mi^2u}(1-2u+u^2)$$
$$=\frac{m+i}{mi^2u}(1-u)^2.$$

So that finally we get, substituting for P and Q their values,
$$\mathrm{A}^{(m)}=\frac{m+i}{mi^2u}(1-u)^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}(1-u);$$
or, since $u=\left(\frac{m}{m+i}\right)^m$,
$$\mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}.$$

If in this equation we make $m=2$, we get for the value of the annuity payable half-yearly
$$\mathrm{A}^{(2)}=\frac{1}{i^2}\left(\frac{2+i}{2}\right)^3\left\{1-\left(\frac{2}{2+i}\right)^2\right\}^2\mathbb{A}+\frac{1}{i}-\frac{2+i}{2i^2}\left\{1-\left(\frac{2}{2+i}\right)^2\right\}$$
$$=\frac{(4i+i^2)^2}{8i^2(2+i)}\mathbb{A}+\frac{1}{i}-\frac{4i+i^2}{2i^2(2+i)}$$
$$=\frac{(4+i)^2}{8(2+i)}\mathbb{A}+\frac{1}{4+2i},$$
which is easily proved to be identical with Davies's result, as given above. The form however thus found is simpler and easier of calculation. The formula for the value of an annuity payable quarterly will be found by making $m=4$. Thus—
$$\mathrm{A}^{(4)}=\frac{1}{i^2}\left(\frac{4+i}{4}\right)^5\left\{1-\left(\frac{4}{4+i}\right)^4\right\}^2\mathbb{A}+\frac{1}{i}-\frac{4+i}{4i^2}\left\{1-\left(\frac{4}{4+i}\right)^4\right\}.$$

This is very much simpler than Davies's formula, with which it is identical, as may be shown by expanding the two, when each of them will be found equal to
$$\frac{65{,}536+49{,}152i+17{,}408i^2+3584i^3+448i^4+32i^5+i^6}{1024(4+i)^3}\times\mathbb{A}$$
$$+\frac{640+160i+20i^2+i^3}{1024}.$$

Mr. Davies finds also the value of $\mathrm{A}^{(m)}$ when m is supposed to become infinite. Resuming the equation,
$$\mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\},$$
we have
$$\left(\frac{m+i}{m}\right)^{m+1}=\frac{m+i}{m}\times\left(\frac{m+i}{m}\right)^m=\frac{m+i}{m}\left(1+\frac{i}{m}\right)^m.$$

Now supposing m to become infinite, the limit of $\dfrac{m+i}{m}$ or $1+\dfrac{i}{m}$ is 1, and the limit of $\left(1+\dfrac{i}{m}\right)^m$ is e^i, e being the base of the Naperian system of logarithms. Substituting then in the value of $A^{(m)}$, its limit, when m becomes infinite, is

$$\frac{1}{i^2}e^i(1-e^{-i})^2\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}(1-e^{-i})$$

$$=\frac{1}{i^2}(e^i-2+e^{-i})\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}(1-e^{-i})$$

$$=\frac{2}{i^2}\left(\frac{i^2}{\lfloor\underline{2}}+\frac{i^4}{\lfloor\underline{4}}+\frac{i^6}{\lfloor\underline{6}}+\ldots\right)\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}\left(i-\frac{i^2}{\lfloor\underline{2}}+\frac{i^3}{\lfloor\underline{3}}-\ldots\right)$$

$$=\left(1+\frac{i^2}{3.4}+\frac{i^4}{3.4.5.6}-\ldots\right)\mathbb{A}+\frac{1}{\lfloor\underline{2}}-\frac{i}{\lfloor\underline{3}}+\frac{i^2}{\lfloor\underline{4}}-\ldots$$

which is the formula given by Davies.

I now proceed to show that Davies's formulæ for the value of an annuity payable half-yearly or quarterly, are identical with Baily's. We have seen that the value of the annuity payable half-yearly is

$$\frac{(4+i)^2}{8(2+i)}\mathbb{A}+\frac{1}{4+2i}.$$

Now let j be the true annual rate of interest, then $1+j=\left(1+\dfrac{i}{2}\right)^2$, whence $i=2\sqrt{1+j}-2$; and substituting for i its value in terms of j, we get the value of an annuity payable half-yearly (writing a_k now for $\mathbb{A}$, since $\mathbb{A}$ is the value of the annuity calculated at the rate of interest j).

$$\frac{(2+2\sqrt{1+j})^2}{16\sqrt{1+j}}a_k+\frac{1}{4\sqrt{1+j}}$$

$$=\frac{2+j+2\sqrt{1+j}}{4\sqrt{1+j}}a_k+\frac{1}{4\sqrt{1+j}}$$

$$=\frac{\sqrt{u}}{4}(2+j+2\sqrt{1+j})a_k+\frac{\sqrt{u}}{4},$$

u being the present value of £1 due in a year at the rate of interest j. This is Baily's formula.

So again, taking the expression for the value of an annuity payable quarterly,

$$\frac{1}{i^2}\left(\frac{4+i}{4}\right)^5\left\{1-\left(\frac{4}{4+i}\right)^4\right\}^2\mathbb{A}+\frac{1}{i}-\frac{4+i}{4i^2}\left\{1-\left(\frac{4}{4+i}\right)^4\right\},$$

we shall now have $1+j=\left(1+\frac{i}{4}\right)^4$, whence $i=4(1+j)^{\frac{1}{4}}-4$, and the expression becomes, substituting as before a_k for $\mathbb{A}$,

$$\frac{(1+j)^{\frac{5}{4}}\left(1-\frac{1}{1+j}\right)^2}{16\{(1+j)^{\frac{1}{4}}-1\}^2}a_k+\frac{1}{4(1+j)^{\frac{1}{4}}-4}-\frac{4(1+j)^{\frac{1}{4}}}{64\{(1+j)^{\frac{1}{4}}-1\}^2}\left(1-\frac{1}{1+j}\right).$$

In order to transform this into the expression given by Baily, put $(1+j)^{\frac{1}{4}}=y$; then it becomes

$$\frac{y^5}{16(y-1)^2}\left(1-\frac{1}{y^4}\right)^2a_k+\frac{1}{4(y-1)}-\frac{y}{16(y-1)^2}\left(1-\frac{1}{y^4}\right)$$

$$=\frac{1}{16y^3}\left(\frac{y^4-1}{y-1}\right)^2a_k+\frac{1}{16y^3(y-1)}\left(4y^3-\frac{y^4-1}{y-1}\right)$$

$$=\frac{(y^3+y^2+y+1)^2}{16y^3}a_k+\frac{4y^3-(y^3+y^2+y+1)}{16y^3(y-1)}$$

$$=\frac{y^6+2y^5+3y^4+4y^3+3y^2+2y+1}{16y^3}a_k+\frac{3y^2+2y-1}{16y^3}$$

$$=\frac{(1+j)^{\frac{6}{4}}+2(1+j)^{\frac{5}{4}}+3(1+j)+4(1+j)^{\frac{3}{4}}+3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}+1}{16(1+j)^{\frac{3}{4}}}\times a_k$$

$$+\frac{3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}-1}{16(1+j)^{\frac{3}{4}}}$$

$$=\frac{u^{\frac{3}{4}}}{16}\{4(1+j)^{\frac{3}{4}}+(4+j)(1+j)^{\frac{1}{2}}+(4+2j)(1+j)^{\frac{1}{4}}+4+3j\}a_k$$

$$+\frac{u^{\frac{3}{4}}}{16}\{3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}-1\},$$

which is Baily's formula.

Resuming now the equation

$$\mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\},$$

put $1+j=\left(1+\frac{i}{m}\right)^m$; then, since $i=m(1+j)^{\frac{1}{m}}-m$, the formula becomes, writing a_k for $\mathbb{A}$, as before,

$$\frac{(1+j)^{\frac{m+1}{m}}}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2}\left\{1-\frac{1}{1+j}\right\}^2a_k+\frac{1}{m\{(1+j)^{\frac{1}{m}}-1\}}$$

$$-\frac{(1+j)^{\frac{1}{m}}}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2}\left\{1-\frac{1}{1+j}\right\}$$

$$=\frac{j^2a_k}{m^2(1+j)^{\frac{m-1}{m}}\{(1+j)^{\frac{1}{m}}-1\}^2}+\frac{1}{m\{(1+j)^{\frac{1}{m}}-1\}}$$

$$-\frac{j}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2(1+j)^{\frac{m-1}{m}}}.$$

Now

$$\frac{1}{(1+j)^{\frac{1}{m}}-1}=\frac{1}{\frac{1}{m}j-\frac{m-1}{2m^2}j^2+\frac{(m-1)(2m-1)}{1.2.3m^3}j^3-\&c.}$$

$$=\frac{m}{j}\,\frac{1}{1-\left(1-\frac{1}{m}\right)\frac{j}{2}+\left(1-\frac{1}{m}\right)\left(2-\frac{1}{m}\right)\frac{j^2}{6}-\&c.}$$

$$=\frac{m}{j}\left\{1+\left(1-\frac{1}{m}\right)\frac{j}{2}-\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}.$$

Also

$$\frac{1}{\{(1+j)^{\frac{1}{m}}-1\}^2}=\frac{m^2}{j^2}\left\{1+\left(1-\frac{1}{m}\right)j+\left(1-\frac{1}{m}\right)\left(1-\frac{5}{m}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\},$$

And

$$(1+j)^{-\frac{m-1}{m}}=1-\frac{m-1}{m}j+\frac{(m-1)(2m-1)}{2m^2}j^2-\cdot\cdot\cdot$$

$$=1-\left(1-\frac{1}{m}\right)j+\left(1-\frac{1}{m}\right)\left(2-\frac{1}{m}\right)\frac{j^2}{2}\cdot\cdot\cdot$$

$$\therefore\quad\frac{1}{(1+j)^{\frac{m-1}{m}}\{(1+j)^{\frac{1}{m}}-1\}^2}=\frac{m^2}{j^2}\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\};$$

and substituting, the value of the annuity becomes

$$\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\cdot\cdot\cdot\right\}a_k$$

$$+\frac{1}{j}\left\{1+\left(1-\frac{1}{m}\right)\frac{j}{2}-\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}$$

$$-\frac{1}{j}\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}$$

$$=\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\cdot\cdot\cdot\right\}a_k+\frac{1}{2}\left(1-\frac{1}{m}\right)-\frac{1}{6}\left(1-\frac{1}{m^2}\right)j+\cdot\cdot\cdot$$

and neglecting terms involving j, the value of the annuity payable m times in the year is $a_k+\frac{1}{2}\left(1-\frac{1}{m}\right)$ or $a_k+\frac{m-1}{2m}$, the well known approximation.

As before, if we wish to obtain the correct value to three decimal places, we must take the next terms, the approximation then being

$$\left(1+\frac{m^2-1}{12m^2}j^2\right)a_k+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}j.$$

In all that precedes it has been supposed that the deaths occurring in any year of age are distributed uniformly throughout the year. This supposition is very generally made by writers on the theory of life contingencies; but although it is very convenient and is very nearly accurate in most cases, it is well known not to be strictly correct, except where the mortality for a series of years agrees with De Moivre's hypothesis, or the numbers dying in successive years of age are equal. If on the other hand, as is generally the case, the numbers dying in successive years of age according to a table of mortality, either increase or decrease as we pass from one age to another, then the supposition fails to give satisfactory results. For if we consider the instantaneous rate of mortality during successive short intervals, we shall find that according to the hypothesis of equal distribution of the deaths, the instantaneous rate of mortality experiences sudden changes in value as we pass from one year to another; and while increasing in value throughout each year, will suddenly become less on entering the next year of age, if the total deaths in that year are less than those in the previous one. This has been pointed out (among other writers) by Mr. Gray in the *Assurance Magazine,* vol. i. pp. 142, 5, and by Professor De Morgan in the *Companion to the Almanac* for 1842. In order to obviate this *discontinuity* in the value of the instantaneous rate of mortality, it will be necessary to suppose the deaths in each year to be distributed over the year in a manner depending on the mortality of the adjacent years. Mr. Gray in the paper already referred to has supposed the deaths in each year to proceed by constant second differences depending on the mortality of the succeeding year; and Professor De Morgan has adopted the still more accurate supposition that the deaths proceed by constant third differences in a manner depending on the mortality both of the preceding and the following years. As an illustration of the extent to which the supposition of uniform distribution may err, I would quote from the Carlisle table the numbers dying at the ages 55 to 60, viz. 73, 76, 82, 93, 106, 122, and again those dying at the ages 80 to 85, viz. 116, 112, 102, 94, 84, 78.

In the next number of the *Journal,* I propose to apply the method of third differences to find more accurately the value of annuities payable half-yearly, &c.

CORRESPONDENCE.

ON CERTAIN FORMULÆ IN MR. DAVID JONES'S WORK ON ANNUITIES.

To the Editor.

Sir,—I am not aware whether what seems to me to be a serious error in one of the expressions given in Jones's work *On Annuities and Reversionary Payments*, for a deferred assurance, has ever been noticed or not.

On page 170, vol. i., of the work referred to, the general value of a deferred assurance on any number of lives is given as

$$\mathrm{A}_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)}\rceil t} = r^{t+1} p_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)},t} - (1-r) a_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)}\rceil t}. \qquad (1)$$

An expression correct enough: but wishing, I presume, to give this result in a more convenient form, the author goes on to manipulate the latter term of the second member of this equation, thus—

$$(1-r) a_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)}\rceil t} = (1-r) r^t p_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)},t} \,.\, a_{\frac{v}{(m+t,\, m_1+t,\, m_2+t,\, \&c.)}};$$

and in this way he deduces

$$\mathrm{A}_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)}\rceil t} = r^t p_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)},t} \, \mathrm{A}_{\frac{v}{(m+t,\, m_1+t,\, m_2+t,\, \&c.)}}. \qquad (2)$$

This latter expression it seems to me is quite erroneous, in consequence of a fatal error in the process that leads to it. In fact, $(1-r) r^t p_{\frac{r}{(m,\, m_1,\, m_2,\, \&c.)},t}$ $.\, a_{\frac{v}{(m+t,m_1+t,m_2+t,\&c.)}}$ is not $\left(\text{except in the single case when } v = m + m_1 + m_2 + \&c. \atop = \textit{number} \text{ of lives}\right)$ the equivalent of $(1-r) a_{\frac{v}{(m,\, m_1,\, m_2,\, \&c.)}\rceil t}$, as assumed in the work referred to.

This will at once be seen if we particularise the expression. Suppose two lives (m and m_1) concerned, and $v=1$; then

$$(1-r) a_{\frac{1}{(m,\, m_1)}\rceil t} = (1-r)(a_{m\rceil t} + a_{m_1\rceil t} - a_{m,m_1\rceil t})$$

$$= r^t(1-r)\left(\frac{l_{m+t}}{l_m} a_{m+t} + \frac{l_{m_1+t}}{l_{m_1}} a_{m_1+t} - \frac{l_{m+t} l_{m_1+t}}{l_m l_{m_1}} a_{m+t,m_1+t}\right),$$

while

$$(1-r) r^t p_{\frac{1}{(m,\, m_1)},t} \, a_{\frac{1}{m+t,m_1+t}} = r^t(1-r)\left(\frac{l_{m+t}}{l_m} + \frac{l_{m_1+t}}{l_{m_1}} - \frac{l_{m+t} l_{m_1+t}}{l_m l_{m_1}}\right)(a_{m+t} + a_{m_1+t} - a_{m+t,m_1+t});$$

a very different expression indeed, and giving rise to a very serious error.

As an illustration, let us take an actual case:—Required the single premium for £1 payable on death of last of two parties now aged 30 and 40 respectively, provided that event happens after 10 years (Carlisle 3 per cent.)

The true value, *i.e.*, the result of equation (1) is ·3118
while the value given by equation (2) is ·2959

a result too little by nearly 2 per cent.

Hoping you will be good enough to inform me whether this point has ever been remarked on previously,

I am, Sir, your most obedient servant,

JAMES R. MACFADYEN.

City of Glasgow Life Assurance Company,
Glasgow, 18th July, 1866.

gram show
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At each Age,
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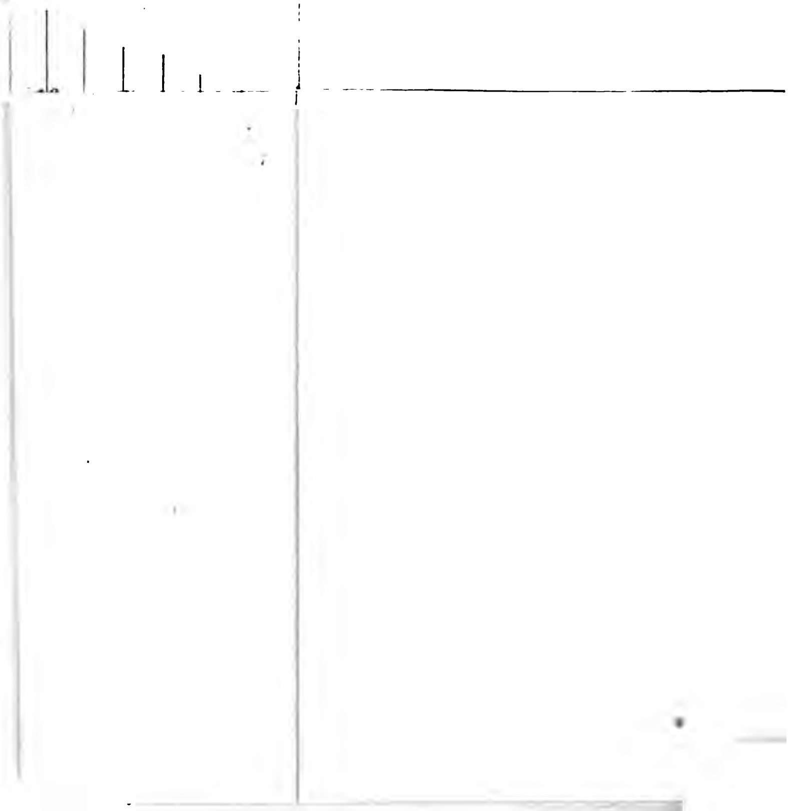

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On the Value of Annuities payable Half-yearly, Quarterly, &c. By T. B. Sprague, M.A., *Actuary of the Equity and Law Life Assurance Society.* (*Part II.*)

[Read before the Institute, December 17th, 1866.]

In the last Number of this *Journal* I examined and compared the solutions of this question given by Milne, Baily, and Griffith Davies in their standard works. The first of these writers cannot be considered to have treated the question in a satisfactory manner. The two latter have investigated the problem more rigorously. They both proceed upon the supposition commonly adopted, viz. that the deaths occurring in each year of age are distributed uniformly over the year. They differ, however, in their views as to the interest to be allowed for fractions of a year, Davies throughout adopting Milne's view, by which the present value of £1 due in six months' time is $\dfrac{1}{1+\dfrac{i}{2}}$, while Baily takes the value as $\dfrac{1}{\sqrt{1+i}}$. The latter, as I have already stated, appears to me the preferable view. Davies, however, has also given an investigation of the value of an annuity payable m times a year, on the supposition that interest is convertible into principal m times in the year; and his solution, as I have shown in the last Number of the *Journal*, is substantially the same as Baily's.

The only authors who have attempted, so far as I am aware, to carry the investigation of the question beyond the point at which it has been left by the above writers, are Professor De Morgan, who gave, without demonstration, a formula in the *Companion to the Almanac* for 1842; and Mr. Woolhouse, who has quite recently given for the first time (*Assurance Magazine*, vol. xi. p. 327) a formula at once very simple and very accurate. The paper of the former is reprinted in the last Number of this *Journal*; and it will be seen upon reference that the formula there given is an exact one—*i.e.* not approximate merely—obtained on the supposition that the deaths in each year proceed by constant third differences, in a manner depending on the mortality of the preceding and following years.* It is, however, far too complicated to be of much use, and it is open to the further objection—a more serious one, in my opinion—that it is based on Milne's supposition as to the interest to be allowed for fractions of a year.

I will now, therefore, proceed to obtain the formula resulting from the adoption of De Morgan's supposition as to the mortality, combined with Baily's supposition as to the interest for fractions of a year. Thus, l_k denoting the number living at the age k, let the number living at the age $k+x$ (where x is less than 1) be

$$l_{k+x}=l_k+Ax+Bx^2+Cx^3 \quad . \quad . \quad . \quad . \quad (1)$$

Here making $x=-1, 1, 2$, successively, in conformity with our supposition that the deaths between the ages k and $k+1$ are to depend on the deaths of the preceding and the following years, we get

$$\begin{aligned} l_{k-1}&=l_k-\ A+\ B-\ C \\ l_{k+1}&=l_k+\ A+\ B+\ C \\ l_{k+2}&=l_k+2A+4B+8C. \end{aligned}$$

From these equations we find

$$\begin{aligned} A&=-\frac{1}{3}l_{k-1}-\frac{1}{2}l_k+l_{k+1}-\frac{1}{6}l_{k+2}, \\ B&=\ \frac{1}{2}l_{k-1}-l_k+\frac{1}{2}l_{k+1}, \\ C&=-\frac{1}{6}l_{k-1}+\frac{1}{2}l_k-\frac{1}{2}l_{k+1}+\frac{1}{6}l_{k+2}. \end{aligned}$$

Then, each year being divided into m equal parts, at the end of each of which a payment $\frac{1}{m}$ is to be made, the value of the payment so made at the end of the xth part of the $(t+1)$th year will be

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* There is in Professor De Morgan's paper what appears to me, if not an error, at least an ambiguity, of language. He says (p. 138), "If *the mortality* of the table be increasing from year to year, it ought to be supposed that the mortality of the latter part of a year is greater than that of the former." The meaning of this is not, as might be supposed at first sight, "If the *rate of mortality* be increasing," &c.; but from the context it appears that the meaning intended is, "If the number dying in a year be increasing from year to year, it ought to be supposed that the number dying in the latter part of the year is greater than the number dying in the former part." I should be glad to learn whether the learned author, on reflection, considers that his words accurately express his meaning.

$$\frac{1}{m}\cdot v^t\cdot\frac{l_{k+t+\frac{x}{m}}}{l_k}\cdot v^{\frac{x}{m}},$$

where the value of $l_{k+t+\frac{x}{m}}$ is to be found in terms of l_{k+t-1}, l_{k+t}, l_{k+t+1}, l_{k+t+2}, by means of the formula (1).

Hence the value of all the payments in the $(t+1)$th year

$$=\frac{v^t}{ml_k}\Sigma_x v^{\frac{x}{m}}\left\{l_{k+t}+\frac{x}{m}\left(-\frac{l_{k+t-1}}{3}-\frac{l_{k+t}}{2}+l_{k+t+1}-\frac{l_{k+t+2}}{6}\right)\right.$$

$$\left.+\frac{x^2}{m^2}\left(\frac{l_{k+t-1}}{2}-l_{k+t}+\frac{l_{k+t+1}}{2}\right)+\frac{x^3}{m^3}\left(-\frac{l_{k+t-1}}{6}+\frac{l_{k+t}}{2}-\frac{l_{k+t+1}}{2}+\frac{l_{k+t+2}}{6}\right)\right\}$$

$$=\frac{v^t}{ml_k}\Sigma_x v^{\frac{x}{m}}\left\{l_{k+t-1}\left(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\right)+l_{k+t}\left(1-\frac{x}{2m}-\frac{x^2}{m^2}+\frac{x^3}{2m^3}\right)\right.$$

$$\left.+l_{k+t+1}\left(\frac{x}{m}+\frac{x^2}{2m^2}-\frac{x^3}{2m^3}\right)+l_{k+t+2}\left(-\frac{x}{6m}+\frac{x^3}{6m^3}\right)\right\},$$

(x having the values 1, 2, 3 . . . m,)

$$=\frac{\text{P}}{m}v^t\cdot\frac{l_{k+t-1}}{l_k}+\frac{\text{Q}}{m}v^t\frac{l_{k+t}}{l_k}+\frac{\text{R}}{m}v^t\frac{l_{k+t+1}}{l_k}+\frac{\text{S}}{m}v^t\frac{l_{k+t+2}}{l_k},\text{ suppose.}$$

And the total value of the annuity will be found by taking the sum of the values obtained by making t equal to 0, 1, 2 . . . . to the extremity of life.

Now

$$\Sigma_t v^t\cdot\frac{l_{k+t-1}}{l_k}=\frac{l_{k-1}}{l_k}+v\cdot\frac{l_k}{l_k}+\frac{v^2l_{k+1}}{l_k}+\frac{v^3l_{k+2}}{l_k}+\text{\&c.}=\frac{l_{k-1}}{l_k}+v(1+a_k)$$

$$\Sigma_t v^t\cdot\frac{l_{k+t}}{l_k}=\frac{l_k}{l_k}+\frac{vl_{k+1}}{l_k}+\frac{v^2l_{k+2}}{l_k}+\ .\ .\ .\ .=1+a_k,$$

$$\Sigma_t v^t\frac{l_{k+t+1}}{l_k}=\frac{l_{k+1}}{l_k}+\frac{vl_{k+2}}{l_k}+\ .\ .\ .\ .=\frac{1}{v}\cdot a_k,$$

$$\Sigma_t v^t\frac{l_{k+t+2}}{l_k}=\frac{l_{k+2}}{l_k}+\frac{vl_{k+3}}{l_k}+\ .\ .\ .\ .$$

$$=-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{1}{v^2}\left(\frac{vl_{k+1}}{l_k}+\frac{v^2l_{k+2}}{l_k}+\frac{v^3l_{k+3}}{l_k}+\ .\ .\ .\right)=-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{a_k}{v^2}.$$

Therefore the value of the annuity is

$$\frac{\text{P}}{m}\left\{\frac{l_{k-1}}{l_k}+v(1+a_k)\right\}+\frac{\text{Q}}{m}(1+a_k)+\frac{\text{R}}{m}\cdot\frac{a_k}{v}+\frac{\text{S}}{m}\left(-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{a_k}{v^2}\right)$$

$$=a_k\left\{\frac{\text{P}v}{m}+\frac{\text{Q}}{m}+\frac{\text{R}}{mv}+\frac{\text{S}}{mv^2}\right\}+\frac{\text{P}}{m}\left(\frac{l_{k-1}}{l_k}+v\right)+\frac{\text{Q}}{m}-\frac{\text{S}}{mv}\frac{l_{k+1}}{l_k}\ .\ .\ .\ (2)$$

We have now to find the values of P, Q, R, S, as defined by the equations

$$P=\Sigma v^{\frac{x}{m}}\left(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\right),\quad Q=\Sigma v^{\frac{x}{m}}\left(1-\frac{x}{2m}-\frac{x^2}{m^2}+\frac{x^3}{2m^3}\right),$$

$$R=\Sigma v^{\frac{x}{m}}\left(\frac{x}{m}+\frac{x^2}{2m^2}-\frac{x^3}{2m^3}\right),\qquad S=\Sigma v^{\frac{x}{m}}\left(-\frac{x}{6m}+\frac{x^3}{6m^3}\right).$$

Put $v^{\frac{1}{m}}=c$; then the following formulæ may be proved for Σxc^x, Σx^2c^x, Σx^3c^x, the summation extending over values of $x=1, 2, 3 \ldots m$.

$$\Sigma(xc^x)=\frac{c-c^{m+1}}{(1-c)^2}-\frac{mc^{m+1}}{1-c},$$

$$\Sigma(x^2c^x)=\frac{(1+c)(c-c^{m+1})}{(1-c)^3}-\frac{2mc^{m+1}}{(1-c)^2}-\frac{m^2c^{m+1}}{1-c},$$

$$\Sigma(x^3c^x)=\frac{(1+4c+c^2)(c-c^{m+1})}{(1-c)^4}-\frac{3mc^{m+1}(1+c)}{(1-c)^3}-\frac{3m^2c^{m+1}}{(1-c)^2}-\frac{m^3c^{m+1}}{1-c}.$$

Then we have

$$P=\Sigma c^x\left(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\right)$$

$$=-\frac{1}{3m}\Sigma(xc^x)+\frac{1}{2m^2}\Sigma(x^2c^x)-\frac{1}{6m^3}\Sigma(x^3c^x).$$

Substituting from above the values of $\Sigma(xc^x)$, &c., we get, after reductions which present no difficulty,

$$P=-\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}+\frac{c(1+c)}{2m^2(1-c)^3}-\frac{2c+c^{m+1}}{6m(1-c)^2}.$$

Similarly

$$Q=\frac{(1+4c+c^2)(c-c^{m+1})}{2m^3(1-c)^4}-\frac{(1+c)(2c+c^{m+1})}{2m^2(1-c)^3}-\frac{c-2c^{m+1}}{2m(1-c)^2}+\frac{c}{1-c},$$

$$R=-\frac{(1+4c+c^2)(c-c^{m+1})}{2m^3(1-c)^4}+\frac{(1+c)(c+2c^{m+1})}{2m^2(1-c)^3}+\frac{2c-c^{m+1}}{2m(1-c)^2}-\frac{c^{m+1}}{1-c},$$

$$S=\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}-\frac{(1+c)c^{m+1}}{2m^2(1-c)^3}-\frac{c+2c^{m+1}}{6m(1-c)^2}.$$

Put $\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}=H$; then, since $c^m=v$, we get

$$\begin{aligned}Pv^3+Qv^2+Rv+S= \;& v^3\left\{-H+\frac{c(1+c)}{2m^2(1-c)^3}-\frac{2c+cv}{6m(1-c)^2}\right\},\\ &+v^2\left\{3H-\frac{(1+c)(2c+cv)}{2m^2(1-c)^3}-\frac{c-2cv}{2m(1-c)^2}+\frac{c}{1-c}\right\}\\ &+v\left\{-3H+\frac{(1+c)(c+2cv)}{2m^2(1-c)^3}+\frac{2c-cv}{2m(1-c)^2}-\frac{cv}{1-c}\right\}\\ &+\;H\;-\frac{(1+c)cv}{2m^2(1-c)^3}-\frac{c+2cv}{6m(1-c)^2}\\ =&\,(1-v)^3H-\frac{c(1-v)^4}{6m(1-c)^2},\end{aligned}$$

and

$$\frac{\mathrm{P}v^3+\mathrm{Q}v^2+\mathrm{R}v+\mathrm{S}}{mv^2}=\frac{(1+4c+c^2)(c-cv)(1-v)^3}{6m^4(1-c)^4v^2}-\frac{c(1-v)^4}{6m^2v^2(1-c)^2}$$

$$=\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}.$$

Substituting in (2), the value of the annuity becomes

$$a_k\cdot\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}$$

$$+\left(\frac{l_{k-1}}{l_k}+v\right)\left\{-\frac{(1+4c+c^2)(c-cv)}{6m^4(1-c)^4}+\frac{c(1+c)}{2m^3(1-c)^3}-\frac{2c+cv}{6m^2(1-c)^2}\right\}$$

$$+\frac{(1+4c+c^2)(c-cv)}{2m^4(1-c)^4}-\frac{(1+c)(2c+cv)}{2m^3(1-c)^3}-\frac{c-2cv}{2m^2(1-c)^2}+\frac{c}{m(1-c)}$$

$$-\frac{l_{k+1}}{l_k}\left\{\frac{(1+4c+c^2)(c-cv)}{6m^4v(1-c)^4}-\frac{cv(1+c)}{2m^3v(1-c)^3}-\frac{c+2cv}{6m^2v(1-c)^2}\right\}$$

$$\left.\begin{aligned}&=\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}\times a_k\\&-\frac{(1+4c+c^2)c(1-v)}{6m^4(1-c)^4}\left\{\frac{l_{k-1}}{l_k}+\frac{1}{v}\frac{l_{k+1}}{l_k}+v-3\right\}+\frac{c(1+c)}{2m^3(1-c)^3}\left\{\frac{l_{k-1}+l_{k+1}}{l_k}-2\right\}\\&-\frac{c}{6m^2(1-c)^2}\left\{(2+v)\frac{l_{k-1}}{l_k}-\frac{1+2v}{v}\cdot\frac{l_{k+1}}{l_k}+3-4v+v^2\right\}+\frac{c}{m(1-c)}\end{aligned}\right\}\ldots(3)$$

This result is exact, according to the suppositions we have made; but it is far too complicated to be of any practical use, in its present form, being considerably more lengthy than Professor De Morgan's formula.

In order to verify the formula, let us apply it to the case of a perpetual annuity certain; for which purpose, we put $l_{k-1}=l_k=l_{k+1}$, and $a_k=\frac{1}{i}$; then it will be found that the formula reduces to $\frac{c}{m(1-c)}=\frac{v^{\frac{1}{m}}}{m(1-v^{\frac{1}{m}})}$, which is the correct value of the annuity on the suppositions we have made.

For $$\frac{1}{m}\left\{v^{\frac{1}{m}}+v^{\frac{2}{m}}+v^{\frac{3}{m}}+\ \ldots\ \text{ad inf.}\right\}=\frac{1}{m}\cdot\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}.$$

The above formula (3) differs from that obtained by Professor De Morgan, for the reason stated above. His formula becomes in the case of the perpetual annuity certain $\frac{1}{i}+\frac{m-1}{2m}$. In order to ascertain how far the two results differ, we must expand the former by powers of i.

Thus since $v=\frac{1}{1+i}$ and $v^{\frac{1}{m}}=(1+i)^{-\frac{1}{m}}$, we have

$$\frac{1}{m}\cdot\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}=\frac{1}{m}\cdot\frac{1}{v^{-\frac{1}{m}}-1}=\frac{1}{m}\cdot\frac{1}{(1+i)^{\frac{1}{m}}-1}$$

$$=\frac{1}{m}\cdot\frac{1}{\frac{1}{m}i-\frac{m-1}{2m^2}i^2+\frac{(m-1)(2m-1)}{6m^3}i^3-\&c.}$$

$$=\frac{1}{i}\cdot\frac{1}{1-\frac{m-1}{2m}i\left\{1-\frac{2m-1}{3m}i+\frac{(2m-1)(3m-1)}{12m^2}i^2-\&c.\right\}}$$

$$=\frac{1}{i}\left[1+\frac{m-1}{2m}i\left\{1-\frac{2m-1}{3m}i+\frac{(2m-1)(3m-1)}{12m^2}i^2-\ \ldots\right\}\right.$$
$$\left.+\frac{(m-1)^2}{4m^2}i^2\left\{1-\frac{2m-1}{3m}i+\ldots\right\}^2+\left(\frac{m-1}{2m}\right)^3 i^3+\ldots\right]$$

$$=\frac{1}{i}+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}i+\frac{m^2-1}{24m^2}i^2-\ \ldots\ \ \ldots \qquad (4)$$

In order to obtain a practical formula from the complicated one above given (3), it will be necessary to expand its terms by powers of i. Before doing this, I propose to examine the formula given by Mr. Woolhouse (*Assurance Magazine*, vol. xi., p. 327),

$$a'_k=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

where μ is the "force of mortality" or the "instantaneous rate of mortality" at the age k, and is equal to the value of $-\frac{1}{l_x}\frac{d}{dx}l_x$, when x is made equal to k; or approximately equal to $\frac{l_{k-1}-l_{k+1}}{2l_k}$; and δ is the "force of discount" and is equal to $-\log_e v$, or $\log_e(1+i)$. Mr. Woolhouse has arrived at this result as a simple case of a problem which he has investigated at great length. It may be readily proved by the following method, which although much shorter than Mr. Woolhouse's, is less elementary. The value of an annuity of £1 payable yearly is

$$a_k=\frac{1}{l_k}\left\{l_{k+1}v+l_{k+2}v^2+\ \ldots\ \ \right\}$$

and that of an annuity of £1 payable by m equal instalments in each year is

$$a'_k=\frac{1}{ml_k}\left\{l_{k+\frac{1}{m}}v^{\frac{1}{m}}+l_{k+\frac{2}{m}}v^{\frac{2}{m}}+\ \ldots\ \right\}$$

Now by a well known theorem proved in the Calculus of Finite Differences, u_x being any function of x,

$$\Sigma u_x=\int u_x dx-\frac{u_x}{2}+\frac{1}{12}\frac{du_x}{dx}-\frac{1}{720}\frac{d^3u_x}{dx^3}+\&c.$$

which between the limits a and b, which are both integers and $b>a$, becomes

$$\Sigma u_b-\Sigma u_a=\int_a^b u_x dx-\frac{u_b}{2}+\frac{1}{12}\left(\frac{du_x}{dx}\right)_b-\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_b+\&c.$$
$$+\frac{u_a}{2}-\frac{1}{12}\left(\frac{du_x}{dx}\right)_a+\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_a-\&c.$$

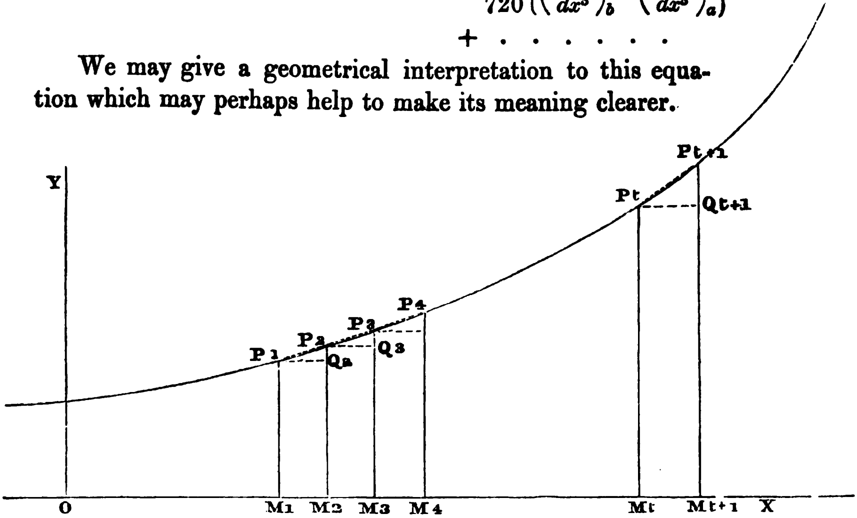
But the first member of this equation is equal to $u_a+u_{a+1}+u_{a+2}+\ldots\ldots+u_{b-1}$, so that we get

$$u_a+u_{a+1}+\ldots\ldots+u_{b-1}=\int_a^b u_x dx$$
$$\left.\begin{array}{l}-\dfrac{u_b}{2}+\dfrac{1}{12}\left(\dfrac{du_x}{dx}\right)_b-\dfrac{1}{720}\left(\dfrac{d^3u_x}{dx^3}\right)_b+\ldots\ldots\\ +\dfrac{u_a}{2}-\dfrac{1}{12}\left(\dfrac{du_x}{dx}\right)_a+\dfrac{1}{720}\left(\dfrac{d^3u_x}{dx^3}\right)_a-\ldots\ldots\end{array}\right\}\ldots(5)$$

We also have

$$\frac{u_a}{2}+u_{a+1}+\ldots\ldots+u_{b-1}+\frac{u_b}{2}=\int_a^b u_x dx+\frac{1}{12}\left\{\left(\frac{du_x}{dx}\right)_b-\left(\frac{du_x}{dx}\right)_a\right\}^*$$
$$-\frac{1}{720}\left\{\left(\frac{d^3u_x}{dx^3}\right)_b-\left(\frac{d^3u_x}{dx^3}\right)_a\right\}$$
$$+\ \ldots\ldots\ldots$$

We may give a geometrical interpretation to this equation which may perhaps help to make its meaning clearer.



Let $OM_1=a$, $OM_{t+1}=b$, so that $OM_{t+1}-OM_1=b-a=t$, suppose; and $M_1M_2=M_2M_3=M_3M_4=\ldots\ldots=M_tM_{t+1}=1$.

Also $P_1M_1=u_a$, $P_2M_2=u_{a+1}$, $\ldots\ldots P_{t+1}M_{t+1}=u_b$; then will the curvilinear area $P_1M_1M_{t+1}P_{t+1}$ be equal to $\int_a^b u_x dx$.

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[*] If the difference between successive values of x be h instead of 1, this formula becomes

$$\frac{u_a}{2}+u_{a+h}+u_{a+2h}+\ldots+u_{b-h}+\frac{u_b}{2}=\frac{1}{h}\int_a^b u_x dx+\frac{h}{12}(u'_b-u'_a)-\frac{h^3}{720}(u'''_b-u'''_a)+\ldots$$

where u'_a, u'''_a, &c., are put for $\left(\frac{du_x}{dx}\right)_a$, $\left(\frac{d^3u_x}{dx^3}\right)_a$, &c.

Compare the formulæ given by Mr. Woolhouse, vol. xi., pp. 317, 331.

Also the area of the rectangle P_1M_2 is equal to $P_1M_1 \times M_1M_2$ $= u_a \times 1 = u_a$; the area of the rectangle P_2M_3 is u_{a+1}; so that $u_a + u_{a+1} + \ldots\ldots + u_{b-1}$ is the sum of the areas of the rectangles P_1M_2, P_2M_3, ... P_tM_{t+1}.

Again the area of the triangle $P_1P_2Q_2$ is equal to $\frac{1}{2} \times P_1Q_2 \times P_2Q_2$ $= \frac{1}{2} \times 1 \times (u_{a+1} - u_a) = \frac{u_{a+1} - u_a}{2}$: the area of the triangle $P_2P_3Q_3$ is $\frac{u_{a+2} - u_{a+1}}{2}$: and so on, till we come to the triangle $P_tP_{t+1}Q_{t+1}$, the area of which is $\frac{u_b - u_{b-1}}{2}$; so that the sum of the areas of the triangles is $\frac{u_b - u_a}{2}$, and the area of the polygon $M_1P_1P_2P_3 \ldots\ldots$ $P_{t+1}M_{t+1}$ is equal to $\frac{u_a}{2} + u_{a+1} + u_{a+2} + \ldots\ldots + u_{b-1} + \frac{u_b}{2}$.

This result may also be arrived at by observing that the areas of the trapeziums, $P_1M_1M_2P_2$, $P_2M_2M_3P_3$, ... $P_tM_tM_{t+1}P_{t+1}$, are $\frac{u_a + u_{a+1}}{2}$, $\frac{u_{a+1} + u_{a+2}}{2}$, ... $\frac{u_{b-1} + u_b}{2}$, respectively.

Thus we see that the sum of the areas of the curvilinear figures P_1P_2, P_2P_3, ... P_tP_{t+1}, is equal to

$$\frac{1}{12}\left\{\left(\frac{du_x}{dx}\right)_b - \left(\frac{du_x}{dx}\right)_a\right\} - \frac{1}{720}\left\{\left(\frac{d^3u_x}{dx^3}\right)_b - \left(\frac{d^3u_x}{dx^3}\right)_a\right\} + \ldots$$

Resuming the equation (5), first write $l_{k+x}v^x$ for u_x, and make $a = k$, and b equal to the extreme age in the table (z). Then the first member of the equation becomes $l_k + l_{k+1}v + l_{k+2}v^2 + \ldots\ldots$, which is equal to $l_k(1 + a_k)$. In the second member we observe that at the extremity of life, the values of u_x, $\frac{du_x}{dx}$, $\frac{d^3u_x}{dx^3}$ .... all vanish, so that we get, the limits of x being 0 and $z - k$,

$$l_k(1 + a_k) = \int_0^{z-k} l_{k+x}v^x dx + \frac{l_k}{2} - \frac{1}{12}\frac{d}{dx}(v^x l_{k+x})_0 + \frac{1}{720}\frac{d^3}{dx^3}(v^x l_{k+x})_0 - \ldots$$

$$\text{or } l_k a_k = P - \frac{l_k}{2} - \frac{Q}{12} + \frac{R}{720} - \ldots\ldots, \text{ suppose.}$$

Next, putting $l_{k+\frac{x}{m}} . v^{\frac{x}{m}}$ for u_x in the equation (5), we get similarly, the limits of x being now 0 and $m(z-k)$

$$l_k\left(\frac{1}{m} + a'_k\right) = \frac{1}{m}\int_0^{m(z-k)} l_{k+\frac{x}{m}} v^{\frac{x}{m}} dx + \frac{l_k}{2m} - \frac{1}{12m}\frac{d}{dx}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0$$
$$+ \frac{1}{720m}\frac{d^3}{dx^3}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 - \ldots\ldots$$

But
$$\int_0^{m(z-k)} l_{k+\frac{x}{m}} v^{\frac{x}{m}} dx = m\int_0^{z-k} l_{k+x} v^x dx = mP$$

$$\frac{d}{dx}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 = \frac{1}{m}\frac{d}{dx}\left(l_{k+x} v^x\right)_0 = \frac{Q}{m}$$

$$\frac{d^3}{dx^3}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 = \frac{1}{m^3}\frac{d^3}{dx^3}\left(l_{k+x} v^x\right)_0 = \frac{R}{m^3}$$

$$\&c. \quad = \quad \&c.$$

whence
$$l_k\left(\frac{1}{m} + a'_k\right) = P + \frac{l_k}{2m} - \frac{Q}{12m^2} + \frac{R}{720m^4} - \&c.$$

and
$$l_k a'_k = P - \frac{l_k}{2m} - \frac{Q}{12m^2} + \frac{R}{720m^4} - \&c.$$

$$\therefore \quad l_k(a'_k - a_k) = \frac{l_k}{2}\cdot\frac{m-1}{m} + \frac{Q}{12}\,\frac{m^2-1}{m^2} - \frac{R}{720}\,\frac{m^4-1}{m^4} + \&c. \ldots (6)$$

It now only remains to find the values of Q, R, &c. Thus, we have

$$\frac{d}{dx}(v^x l_{k+x}) = v^x \log_e v\, l_{k+x} + v^x \frac{d}{dx} l_{k+x}$$

and putting $x=0$ we get

$$\begin{aligned} Q &= \log_e v \,.\, l_k + \left(\frac{d}{dx} l_{k+x}\right)_0 \\ &= -l_k \log_e(1+i) - \mu l_k, \text{ since } \mu = -\frac{1}{l_k}\cdot\left(\frac{d}{dx} l_{k+x}\right)_0 \\ &= -l_k(\mu+\delta) \end{aligned}$$

Also

$$\frac{d^3}{dx^3}(v^x l_{k+x}) = v^x \log^3 v\, l_{k+x} + 3v^x \log^2 v \frac{d}{dx} l_{k+x} + 3v^x \log v \frac{d^2}{dx^2} l_{k+x} + v^x \frac{d^3}{dx^3} l_{k+x}$$

whence, making $x=0$, as before,

$$\begin{aligned} R &= \log^3 v\, l_k + 3\log^2 v\left(\frac{d}{dx} l_{k+x}\right)_0 + 3\log v\left(\frac{d^2}{dx^2} l_{k+x}\right)_0 + \left(\frac{d^3}{dx^3} l_{k+x}\right)_0 \\ &= -\delta^3 l_k - 3\delta^2 \mu l_k - 3\delta l''_k + l'''_k. \end{aligned}$$

Substituting in (6), and dividing by l_k, we get

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta) + \frac{m^4-1}{720m^4}\left(\delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k}\right) - \&c.$$

Here μ is approximately equal to $\frac{l_{k-1} - l_{k+1}}{2l_k}$. Its value, as well as those of l''_k and l'''_k, may be found more exactly as follows. Let the tabulated values of l_x adjoining l_k be differenced, as is done by Mr. Woolhouse (*Assurance Magazine*, vol. xi. p. 64), then by the formula which he has given on p. 68, we have

$$\begin{aligned} l_{k+x}=l_k+&\left(a_0-\frac{c_0}{6}+\frac{e_0}{30}-\frac{g_0}{140}\right)x \\ &+\left(\frac{b_0}{2}-\frac{d_0}{24}+\frac{f_0}{180}-\frac{h_0}{1120}\right)x^2 \\ &+\left(\frac{c_0}{6}-\frac{e_0}{24}+\frac{7g_0}{720}\right)x^3+\&c. \end{aligned}$$

Hence $$\left(\frac{d}{dx}l_{k+x}\right)_0=-\mu l_k=a_0-\frac{c_0}{6}+\frac{e_0}{30}-\frac{g_0}{140}$$

$$l''=\left(\frac{d^2}{dx^2}l_{k+x}\right)_0=2\left(\frac{b_0}{2}-\frac{d_0}{24}+\frac{f_0}{180}-\frac{h_0}{1120}\right)$$

$$l'''_k=\left(\frac{d^3}{dx^3}l_{k+x}\right)_0=6\left(\frac{c_0}{6}-\frac{e_0}{24}+\frac{7g_0}{720}\right)$$

In order to estimate the magnitude of the various terms in the above expression for a'_k, let us take a numerical example; and since the importance of the corrections introduced by the third and fourth terms will be greatest at advanced ages, let us suppose $k=85$. Then, using the "Experience" table, we have

Age.	Number Living.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
(x)	(l_x)	$(-d_x)$				
80	13290					
		−1866				
81	11424		+136			
		−1730		+12		
82	9694		148		−5	
		−1582		7		+2
83	8112		155		−3	
		−1427		4		−3
84	6685		159		−6	
		−1268		− 2		+4
85	5417	· · · ·	157	· · ·	−2	· · ·
		−1111		− 4		0
86	4306		153		−2	
		− 958		− 6		−1
87	3348		147		−3	
		− 811		− 9		+2
88	2537		138		−1	
		− 673		−10		
89	1864		128			
		− 545				
90	1319					

Here $$a_0=\tfrac{1}{2}(-1268-1111)=-\frac{2379}{2}=-1189{\cdot}5$$

$$b_0=157$$

$$c_0=\tfrac{1}{2}(-2-4)=-3$$

$$d_0=-2$$

$$e_0=\tfrac{1}{2}(4+0)=2$$

Also supposing the rate of interest to be 3 per cent. we have $\delta=\log_e(1+i)=\log_e(1{\cdot}03)={\cdot}02956$.

Hence we find

$$\mu = \cdot 21948,\ \frac{l''_k}{l_k} = \frac{157 \cdot 167}{5417} = \cdot 029014,\ \frac{l'''_k}{l_k} = -\frac{3 \cdot 5}{5417} = -\cdot 000646$$

$$\begin{aligned} \delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k} &= \cdot 000026 \\ &\quad + \cdot 000575 \\ &\quad + \cdot 002573 \\ &\quad + \cdot 000646 \\ &= \cdot 003820 \end{aligned}$$

and

$$\frac{1}{720}\left(\delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k}\right) = \cdot 000005$$

We hence conclude that, omitting the term involving this quantity, and the following terms, the value of a'_k may be found correct to five decimal places from the equation

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu + \delta) \quad \ldots\ldots \quad (7)$$

Again

$$\begin{aligned} \mu + \delta &= \cdot 21948 + \cdot 02956 \\ &= \cdot 24905 \end{aligned}$$

and

$$\frac{\mu + \delta}{12} = \cdot 02075\,;$$

whence we conclude that the common approximation $a_k + \frac{m-1}{2m}$ is only true to the first decimal place in this instance.

Lastly, if we take the approximate value of μ, viz. $-\frac{a_0}{l_k}$ $(= \cdot 21958)$, the error in the value of the annuity is equal to $-\frac{m^2-1}{12m^2} \times \cdot 000105$, and will not amount to more than a unit in the fifth decimal place. For any purpose that is ever likely to be required, we may therefore use the formula

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left\{\frac{l_{k-1} - l_{k+1}}{2l_k} + \log_e(1+i)\right\}$$

For younger ages the last term of this equation will be much less important, but it will be found in all cases to affect the third place of decimals. It will also be noticed that the value of the third term becomes slightly greater as the rate of interest is greater.

The preceding expression for a'_k may be usefully compared with the one found above (4) for the value of a perpetual annuity certain. Thus, putting $l_{k-1} = l_{k+1}$, and $a_k = \frac{1}{i}$, it becomes

$$\frac{1}{i} + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left\{i - \frac{i^2}{2} + \ldots\ldots\right\}$$

which agrees with (4) as far as the term involving i^2.

Again, at 3 per cent. interest $i = \cdot 03$, $\delta = \cdot 02956$ $\therefore i - \delta = \cdot 00044$ and $\frac{m^2-1}{12m^2}(i-\delta) < 00004$, so that if we use the formula

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k} + i\right) \ldots . \quad (8)$$

we may expect to obtain the value of a'_k correct to four places of decimals.

In this equation making $m = 2, 4, \infty$, in succession, and putting μ for its approximate value $\frac{l_{k-1}-l_{k+1}}{2l_k}$, we have the value of an annuity of £1 payable half yearly equal to $a_k + \frac{1}{4} - \frac{1}{16}(\mu + i)$

quarterly „ $a_k + \frac{3}{8} - \frac{5}{64}(\mu + i)$

momently „ $a_k + \frac{1}{2} - \frac{1}{12}(\mu + i)$

Now, returning to the equation (3), our object will be to expand it in a series of ascending powers of i.

We have

$$c = v^{\frac{1}{m}} = \frac{1}{(1+i)^{\frac{1}{m}}} = (1+i)^{-\frac{1}{m}}$$

or

$$c = 1 - \frac{i}{m} + \frac{m+1}{2m^2}i^2 - \frac{(m+1)(2m+1)}{6m^3}i^3 + \frac{(m+1)(2m+1)(3m+1)}{24m^4}i^4 - \text{\&c.}$$

$$\therefore \frac{1}{m(1-c)} = \frac{1}{i\left\{1 - \frac{m+1}{2m}i + \frac{(m+1)(2m+1)}{6m^2}i^2 - \frac{(m+1)(2m+1)(3m+1)}{24m^3}i^3 + \ldots\right\}}$$

$$= \frac{1}{i}\,\frac{1}{1 - \frac{m+1}{2m}i\left\{1 - \frac{2m+1}{3m}i + \frac{(2m+1)(3m+1)}{12m^2}i^2 \ldots .\right\}}$$

Hence

$$\frac{1}{m^t(1-c)^t} = \frac{1}{i^t}\left[1 - \frac{m+1}{2m}i\left\{1 - \frac{2m+1}{3m}i + \frac{(2m+1)(3m+1)}{12m^2}i^2 - \ldots\right\}\right]^{-t}$$

$$= \frac{1}{i^t}\left[1 + t\cdot\frac{m+1}{2m}i\left\{1 - \frac{2m+1}{3m}i + \frac{(2m+1)(3m+1)}{12m^2}i^2 - \ldots\right\}\right.$$

$$\left. + \frac{t(t+1)}{1.2}\,\frac{(m+1)^2}{4m^2}i^2\left\{1 - \frac{2m+1}{3m}i + \ldots\right\}^2 + \frac{t(t+1)(t+2)}{6}\,\frac{(m+1)^3}{8m^3}i^3 + \ldots .\right]$$

$$=\frac{1}{i^t}\Big[1+t\frac{m+1}{2m}i+t\cdot\frac{m+1}{2m^2}i^2\left\{-\frac{2m+1}{3}+\frac{t+1}{2}\frac{m+1}{2}\right\}$$

$$+t\cdot\frac{m+1}{2m^3}i^3\left\{\frac{(2m+1)(3m+1)}{12}-\frac{(t+1)(m+1)(2m+1)}{12}\right.$$

$$\left.+\frac{(t+1)(t+2)(m+1)^2}{24}\right\}+\ldots\ldots\Big]$$

$$=\frac{1}{i^t}\left\{1+t\frac{m+1}{2m}i+t\frac{m+1}{24m^2}(\overline{3t-5}\,.m+3t-1)i^2\right.$$

$$\left.+t\frac{m+1}{48m^3}[(t^2-t+10)m^2+2(t^2+4)m+t^2+t+2]i^3+\ldots\right\}$$

Now making $t=1, 2, 3, 4$, successively

$$\frac{1}{m(1-c)}=\frac{1}{i}\left\{1+\frac{m+1}{2m}i-\frac{m^2-1}{12m^2}i^2+\frac{(m+1)(5m^2+5m+2)}{24m^3}i^3-\&c.\right\}$$

$$\frac{1}{m^2(1-c)^2}=\frac{1}{i^2}\left\{1+\frac{m+1}{m}i+\frac{(m+1)(m+5)}{12m^2}i^2\right.$$

$$\left.+\frac{(m+1)(3m^2+4m+2)}{6m^3}i^3+\ldots\right\}$$

$$\frac{1}{m^3(1-c)^3}=\frac{1}{i^3}\left\{1+\frac{3}{2}\frac{m+1}{m}i+\frac{(m+1)(m+2)}{2m^2}i^2\right.$$

$$\left.+\frac{(m+1)(16m^2+26m+15)}{16m^3}i^3+\ldots\right\}$$

$$\frac{1}{m^4(1-c)^4}=\frac{1}{i^4}\left\{1+2\frac{m+1}{m}i+\frac{(m+1)(7m+11)}{6m^2}i^2\right.$$

$$\left.+\frac{(m+1)(11m^2+20m+11)}{6m^3}i^3+\ldots\right\}$$

Also $\dfrac{c(1-v)^4}{v^2}=\dfrac{v^{\frac{1}{m}}(1-v)^4}{v^2}=\dfrac{i^4}{(1+i)^{2+\frac{1}{m}}}$

$$=i^4(1+i)^{-\frac{2m+1}{m}}$$

$$=i^4\left\{1-\frac{2m+1}{m}i+\frac{(2m+1)(3m+1)}{2m^2}i^2\right.$$

$$\left.-\frac{(2m+1)(3m+1)(4m+1)}{6m^3}i^3+\ldots\right\}$$

and $\quad c^2=(1+i)^{-\frac{2}{m}}$

$$=1-\frac{2}{m}i+\frac{2+m}{m^2}i^2-\frac{2(2+m)(1+m)}{3m^3}i^3+\ldots$$

and $\quad 4c=4-\dfrac{4i}{m}+\dfrac{2(m+1)}{m^2}i^2-\dfrac{2}{3}\dfrac{(m+1)(2m+1)}{m^3}i^3+\ldots$

$$\therefore 1+4c+c^2=6-\frac{6i}{m}+\frac{3m+4}{m^2}i^2-\frac{2(m+1)^2}{m^3}i^3+\ldots.$$

Again,

$$m(1-c)=i\left\{1-\frac{m+1}{2m}i+\frac{(m+1)(2m+1)}{6m^2}i^2\right.$$
$$\left.-\frac{\overline{m+1}.\overline{2m+1}.\overline{3m+1}}{24m^3}i^3\right\}$$

$$m^2(1-c)^2=i^2\left\{1-\frac{m+1}{m}i+\frac{(m+1)(11m+7)}{12m^2}i^2+\ldots.\right\}$$

$$\therefore 1+4c+c^2-m^2(1-c)^2=6-\frac{6i}{m}-\frac{m^2-3m-4}{m^2}i^2$$
$$+\frac{(m+1)(m^2-2m-2)}{m^3}i^3-\ldots.$$

Substituting these values, the coefficient of a_k in (3) becomes therefore

$$\left\{1-\frac{i}{m}-\frac{m^2-3m-4}{6m^2}i^2+\frac{\overline{m+1}.\overline{m^2-2m-2}}{6m^3}i^3-\ldots.\right\}$$
$$\times\left\{1-\frac{2m+1}{m}i+\frac{\overline{2m+1}.\overline{3m+1}}{2m^2}i^2-\frac{\overline{2m+1}.\overline{3m+1}.\overline{4m+1}}{6m^3}i^3+\ldots\right\}$$
$$\times\left\{1+2\times\frac{m+1}{m}i+\frac{\overline{m+1}.\overline{7m+11}}{6m^2}i^2+\frac{\overline{m+1}.\overline{11m^2+20m+11}}{6m^3}i^3+\ldots\right\}$$
$$=1\ldots\ldots\ldots+\frac{10m^3-25m^2+20m+5}{6m^3}i^3+\ldots.$$
$$=1+\frac{5}{6}\frac{(2m+1)(m+1)^2}{m^3}i^3+\ldots.$$

Thus we see that the coefficient of a_k does not contain any terms involving the first and second powers of i; and it seems a fair inference that if we found in the same way the value of the annuity on the supposition that the deaths in each year were distributed according to constant *fifth* differences, then the term involving i^3 would also disappear from the coefficient of a_k; and that further terms would disappear from the coefficient if the approximation were continued further, since in the formula obtained by the more correct method given by Mr. Woolhouse, the coefficient of a_k is simply unity.

We may obtain in the same way the values of the other terms in (3), but the following method is more simple. Put $1-c=y$, then the coefficient of $\frac{l_{k-1}}{l_k}+v$ becomes, since $c=1-y$,

$$\{-(1+4c+c^2)(c-c^{m+1})+3mc(1-c^2)-m^2(1-c)^2(2c+c^{m+1})\}\div 6m^4(1-c)^4$$

$$=\frac{c}{6m^4y^4}\left[-(6-6y+y^2)\{1-(1-y)^m\}+3m(2y-y^2)-m^2y^2\{2+(1-y)^m\}\right]$$

$$=\frac{c}{6m^4y^4}\left[-6+6(m+1)y-(2m^2+3m+1)y^2+\{6-6y-(m^2-1)y^2\}(1-y)^m\right]$$

$$=\frac{c}{6m^4y^4}\left[-6+6(m+1)y-(2m^2+3m+1)y^2+\{6-6y-(m^2-1)y^2\}\left\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{24}y^4..\right\}\right]$$

$$=\frac{c}{6m^4y^4}\Big[-6+(6m+1)y-(2m^2+3m+1)y^2+6-6my+3m.\overline{m-1}.y^2-m.\overline{m-1}.\overline{m-2}.y^3+\frac{1}{4}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.y^4$$

$$-\frac{1}{20}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}.y^5....+(-1)^r6\cdot\frac{m.\overline{m-1}.\overline{m-2}....\overline{m-r+1}}{\underline{|r}}y^r+....$$

$$-6y+6my^2-3m.\overline{m-1}y^3+m.\overline{m-1}.\overline{m-2}.y^4-\frac{1}{4}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.y^5....+(-1)^r6\cdot\frac{m.\overline{m-1}....\overline{m-r+2}}{\underline{|r-1}}y^r+....$$

$$-(m^2-1)y^2+m(m^2-1)y^3-\frac{1}{2}m.\overline{m-1}.(m^2-1)y^4+\frac{1}{6}m.\overline{m-1}.\overline{m-2}(m^2-1)y^5....-(-1)^r\frac{(m^2-1)m.\overline{m-1}....\overline{m-r+3}}{\underline{|r-2}}y^r+....$$

$$=\frac{c}{6m^4y^4}\left[-\frac{m^2(m^2-1)}{6}y^4+\frac{m.\overline{m-1}.\overline{m-2}}{60}(7m^2+6m-1)y^5+...+(-1)^r\frac{m.\overline{m-1}...\overline{m-r+3}}{\underline{|r}}\times(r-3)\{(r+2)m^2+6m-r-4\}y^r+...\right]$$

$$=\frac{1-y}{6m^3}\left[-\frac{m(m^2-1)}{4}+\frac{\overline{m-1}.\overline{m-2}}{60}(7m^2+6m-1)y....+(-1)^r\frac{\overline{m-1}.\overline{m-2}....\overline{m-r+3}}{\underline{|r}}\times(r-3)\{(r+2)m^2+6m-r-4\}y^{r-4}+...\right]$$

$$=\frac{1}{6m^3}\left[-\frac{m.(m^2-1)}{4}+\frac{(m^2-1)(7m^2+2)}{60}y+\ldots.\right]$$

$$=-\frac{m^2-1}{24m^2}+\frac{(m-1)(7m^2+2)}{360m^4}i,\text{ approximately, since }y=1-c=1-(1+i)^{-\frac{1}{m}}=\frac{i}{m}-\frac{m+1}{2m^2}i^2+\ldots.$$

The coefficient of $-\frac{l_{k+1}}{l_k}$ becomes $\frac{c}{6m^4v(1-c)^4}\{(1+4c+c^2)(1-c^m)-3mc^m(1-c^2)-m^2(1+2c^m)(1-c)^2\}$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[(6-6y+y^2)\{1-(1-y)^m\}-3m(1-y)^m(2y-y^2)-m^2y^2\{1+2(1-y)^m\}\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[6-6y-(m^2-1)y^2-\{6-6y+y^2+6my-3my^2+2m^2y^2\}(1-y)^m\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[6-6y-(m^2-1)y^2-\{6+6(m-1)y+(2m^2-3m+1)y^2\}\left\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{24}y^4+\ldots\right\}\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[6-6y-(m^2-1)y^2-6+6my-3m.\overline{m-1}.y^2+m.\overline{m-1}.\overline{m-2}.y^3-\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{4}.y^4+\frac{m.\overline{m-1}\ldots\overline{m-4}}{20}y^5-\ldots\right.$$

$$+(-1)^r6\frac{m.\overline{m-1}\ldots\overline{m-r+1}}{\underline{|r}}y^r+\ldots$$

$$-6(m-1)y+6m(m-1)y^2-3m(m-1)^2y^3+(m-1)m.(m-1)(m-2)y^4-\frac{m.\overline{m-1}\ldots\overline{m-3}(m-1)}{4}y^5+\ldots$$

$$-(-1)^r6(m-1)\frac{m.\overline{m-1}\ldots\overline{m-r+2}}{\underline{|r-1}}y^r+\ldots$$

$$-(2m^2-3m+1)y^2+m(2m^2-3m+1)y^3-(m-1)(2m-1)\frac{m.\overline{m-1}}{2}y^4+(m-1)(2m-1)\frac{m.\overline{m-1}.\overline{m-2}}{6}y^5-\ldots.$$

$$+(-1)^r(2m^2-3m+1)\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r-2}}y^r+\ldots\Bigg]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[-\frac{m^2(m^2-1)}{4}y^4+\frac{m.\overline{m-1}.\overline{m-2}}{60}(8m^2+9m+1)y^5-\ldots-(-1)^r\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r}}(r-3)\{2m^2(r-1)+3m(r-2)+r-4\}y^r+\ldots\right]$$

$$=\frac{1}{6c^{m-1}}\left[-\frac{m^2-1}{4m^2}+\frac{\overline{m-1}.\overline{m-2}.(8m^2+9m+1)}{60m^3}y-\ldots\right]$$

$$=\frac{1}{6}(1+i)^{\frac{m-1}{m}}\left[\qquad\qquad\right]$$

$$=\frac{1}{6}\left[1+\frac{m-1}{m}i-\frac{m-1}{2m^2}i^2+\ldots\right]\left[-\frac{m^2-1}{4m^2}+\frac{\overline{m-1}.\overline{m-2}.(8m^2+9m+1)}{60m^4}i-\ldots.\right]$$

$$=\frac{1}{6}\left[-\frac{m^2-1}{4m^2}-\frac{(m^2-1)(7m^2+2)}{60m^4}i-\ldots.\right.$$

$$=-\frac{m^2-1}{24m^2}-\frac{(m^2-1)(7m^2+2)}{360m^4}i+\ldots.$$

The other term becomes similarly

$$\{(1+4c+c^2)(c-c^{m+1})-m(1-c^2)(2c+c^{m+1})-m^2(1-c)^2(c-2c^{m+1})+2m^3c(1-c)^3\}\div 2m^4(1-c)^4$$

$$=\frac{c}{2m^4(1-c)^4}\{(1+4c+c^2)(1-c^m)-m(1-c^2)(2+c^m)-m^2(1-c)^2(1-2c^m)+2m^3(1-c)^3\}$$

$$=\frac{c}{2m^4y^4}\{(6-6y+y^2)(1-c^m)-m(2y-y^2)(2+c^m)-m^2y^2(1-2c^m)+2m^3y^3\}$$

$$=\frac{c}{2m^4y^4}\left[6-6y+y^2-4my+2my^2-m^2y^2+2m^3y^3-(1-y)^m\{6-6y+y^2+2my-my^2-2m^2y^2\}\right]$$

$$=\frac{c}{2m^4y^4}\Big[6-(6+4m)y-(m^2-2m-1)y^2+2m^3y^3-\Big\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\ldots\Big\}\{6+(2m-6)y-(2m^2+m-1)y^2\}\Big]$$

$$=\frac{c}{2m^4y^4}\Big[6-(4m+6)y-(m^2-2m-1)y^2+2m^3y^3-6+6my-3m.\overline{m-1}.y^2+m.\overline{m-1}.\overline{m-2}y^3-\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{4}y^4$$

$$+\frac{m.\overline{m-1}\ldots\overline{m-4}}{20}y^5+\ldots+(-1)^r6\cdot\frac{m.\overline{m-1}\ldots\overline{m-r+1}}{\underline{|r}}y^r+\ldots$$

$$-(2m-6)y+m(2m-6)y^2-m.\overline{m-1}.\overline{m-3}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{3}y^4-\frac{m.\overline{m-1}.\overline{m-2}.(m-3)^2}{12}y^5+\ldots$$

$$-(-1)^r(2m-6)\frac{m.\overline{m-1}\ldots\overline{m-r+2}}{\underline{|r-1}}y^r+\ldots$$

$$+(2m^2+m-1)y^2-m(2m^2+m-1)y^3+\frac{m.\overline{m-1}}{2}(2m^2+m-1)y^4-\frac{m.\overline{m-1}.\overline{m-2}(2m^2+m-1)}{6}y^5+\ldots$$

$$-(-1)^r(2m^2+m-1)\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r-2}}y^r+\ldots\Big]$$

$$=\frac{c}{2m^4y^4}\Big[\frac{m.\overline{m-1}}{12}(13m^2+m)y^4-\frac{m.\overline{m-1}.\overline{m-2}}{60}(22m^2+m-1)y^5\ldots.$$

$$+(-1)^r\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r}}\{2(r^2-3)m^2-(r-3)(r-6)m-(r-3)(r-4)\}y^r+\ldots.\Big]$$

$$=\frac{1-y}{2}\Big[\frac{(m-1)(13m+1)}{12m^2}-\frac{(m-1)(m-2)(22m^2+m-1)}{60m^3}y+\ldots\Big]$$

$$=\frac{(m-1)(13m+1)}{24m^2}-\frac{m^2-1}{60m^3}(11m^2+1)y+\ldots$$

$$=\frac{(m-1)(13m+1)}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}i,\text{ approximately.}$$

Now substituting these values in (3), the value of the correction becomes

$$\left\{\frac{l_{k-1}}{l_k}+1-i+i^2-i^3+\ldots\right\}\left\{-\frac{m^2-1}{24m^2}+\frac{(m^2-1)(7m^2+2)}{360m^4}i-\ldots\right\}$$

$$+\frac{l_{k+1}}{l_k}\left\{\frac{m^2-1}{24m^2}+\frac{(m^2-1)(7m^2+2)}{360m^4}i-\ldots\right\}$$

$$+\frac{(m-1)(13m+1)}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}i+\ldots$$

$$=\frac{m-1}{2m}-\frac{m^2-1}{24m^2}\cdot\frac{l_{k-1}-l_{k+1}}{l_k}$$

$$-\left\{\frac{m^2-1}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}+\frac{(m^2-1)(7m^2+2)}{360m^4}\left(\frac{l_{k-1}+l_{k+1}}{l_k}+1\right)\right\}i, \text{ approximately.}$$

$$=\frac{m-1}{2m}-\frac{m^2-1}{24m^2}\cdot\frac{l_{k-1}-l_{k+1}}{l_k}+\frac{m^2-1}{12m^2}\left\{-\frac{2(11m^2+1)}{15m^2}+\frac{7m^2+2}{30m^2}\cdot\frac{l_{k-1}+l_{k+1}}{l_k}\right\}i.$$

But $\frac{l_{k-1}+l_{k+1}}{l_k}=2$, very nearly; and making this substitution, the value of the correction becomes

$$\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)$$

which agrees with the formula (8) we have found above, and is substantially the same as Mr. Woolhouse's formula (7).

The error caused by putting 2 for $\frac{l_{k-1}+l_{k+1}}{l_k}$ is equal to $\frac{l_{k-1}+l_{k+1}}{l_k}-2=\frac{l_{k-1}-l_k-(l_k-l_{k+1})}{l_k}=\frac{d_{k-1}-d_k}{l_k}$. Taking as before the Experience table of mortality, and making $k=85$, the value of this fraction is $\frac{157}{5417}$, or ·03 very nearly; and the value will be generally much less. This error has to be multiplied into $\frac{m^2-1}{12m^2}\cdot\frac{7m^2+2}{30m^2}\cdot i$, and the value of this quantity, making $m=2$, $i=\cdot03$, *i.e.* taking the case of an annuity payable half-yearly at 3 per cent. interest, is $\frac{3}{48}\cdot\frac{30}{120}\times\cdot03=\frac{\cdot03}{64}$, so that the amount of the error is nearly $\frac{\cdot0009}{64}$ or ·000014, and is therefore quite insignificant.

We have thus a very exact correspondence between the results obtained by the supposition that the deaths are distributed in each year according to constant third differences, and by the more accurate method given by Mr. Woolhouse; and the problem of finding the value of a life annuity payable by m equal instalments may be considered as completely solved.

I purpose considering on a future occasion several allied questions—such as (1) the increase in the value of an annuity payable m times a year, when a proportionate part is paid to the day of death, (2) the value of an assurance payable at the instant of death, or at an assigned date—three or six months after death—(3) the value of a survivorship assurance, subject to the same modifications, (4) the value of a reversionary annuity which is to run from the date of the death of the life assured, the first payment being made twelve months after death instead of, as usually supposed, at the end of the year in which death occurs, (5) the value of the same annuity when payable half-yearly, quarterly, &c., or when a proportionate part is paid up to the date of death of the nominee. In these and other cases the usual formulæ are not strictly applicable to the questions which occur in practice; and it will be satisfactory, and may occasionally be useful, to know how to calculate the values more accurately.

For facility of reference I subjoin a list of the formulæ discussed in the preceding paper—the formulæ being first set down as given by the respective authors, and the modifications thereof made by myself being marked (*).

Milne, $a_k^{(m)} = a_k + \dfrac{m-1}{2m}$.

Baily, $$a_k^{(2)} = \frac{\sqrt{v}}{4}\{2(1+i)^{\frac{1}{2}} + 2 + i\}a_k + \frac{\sqrt{v}}{4}$$

$$= \left(1 + \frac{i^2}{16} - \frac{i^3}{16} + \ldots\right)a_k + \frac{1}{4} - \frac{i}{8} + \frac{3}{32}i^2 + \ldots *$$

„ $$a_k^{(4)} = \frac{v^{\frac{3}{4}}}{16}\{4(1+i)^{\frac{3}{4}} + (4+i)(1+i)^{\frac{1}{2}} + 2(2+i)(1+i)^{\frac{1}{4}} + 4 + 3i\}a_k$$

$$+ \frac{v^{\frac{3}{4}}}{16}\{3(1+i)^{\frac{1}{2}} + 2(1+i)^{\frac{1}{4}} + 1\}$$

$$= \left\{1 + \frac{5}{16}i^2 - \ldots\right\}a_k + \frac{3}{8} - \frac{5}{32}i + \frac{15}{128}i^2 - \ldots *$$

Griffith Davies, $A^{(m)} = (P+Q)\mathbb{A} + P$,

where $$P = \Sigma_x \frac{(m-x)m^{x-2}}{(m+i)^x}, \quad Q = \left(1 + \frac{i}{m}\right)^m \Sigma_x \frac{xm^{x-2}}{(m+i)^x},$$

and $\mathbb{A}$ is the value of an annuity of £1 payable yearly, calculated at the true annual rate of interest $\left(1+\frac{i}{m}\right)^m-1=j$;

$$\text{or } \mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^*$$

$$\text{or } a_k^{(m)}=\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\ldots\ldots\right\}a_k+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}j+\ldots\ldots^*$$

In the last three formulæ, but in none of the others here given, i is the nominal rate of interest convertible m times a year, and j is the true annual interest.

$$\text{Woolhouse, } a_k^{(m)}=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

$$=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)^*$$

Errata in Part I.

Page 190, line 3 from bottom, for $\frac{3}{32}i^3$ read $\frac{3}{32}i^2$.
„ 192, „ 19, for $(m+1)^x$ read $(m+i)^x$.
„ 193, „ 16, for Σ^t read Σ_t.
„ „ throughout, for p_{k+t} read $p_{k,t}$.
„ „ „ for p_{k+t-1} read $p_{k,t-1}$.
„ „ line 20, for "value" read "amount".
„ 195, „ 6 from bottom, for $\frac{640+160i+20i^2+i^3}{1024}$ read $\frac{96+32i+3i^2}{4(4+i)^3}$.

On Annuities and Assurances on Successive Lives. By THOMAS WEDDLE, F.R.A.S.*

MILNE, in his *Treatise on Annuities*, chapter vii., has discussed this subject with much clearness and elegance; yet his method of investigation does not seem to be the best possible, nor do I think that his results are given in forms well adapted to numerical applications. It may therefore be not altogether useless to consider the subject in a somewhat different manner, and so to exhibit the formulas as best to meet the wants of the computer.

PROBLEM I.—To determine the present value (a_n) of an assurance of £1 payable on the failure of the last of n successive lives $A', A'' \ldots\ldots A^{(n)}$.

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* This paper appeared in the *Philosophical Magazine* for January, 1850, and is the one referred to by Mr. Peter Gray, at page 1, vol. ii., of this *Journal.*—ED. *J. I. A.*

Let $\mathrm{a}^{(p)}$ denote the value at the time of nomination of an assurance of £1 payable on the death of $\mathrm{A}^{(p)}$; $\mathrm{A}^{(p)}$ itself denoting the value of an annuity of £1 during the life of $\mathrm{A}^{(p)}$, so that we have (r being the interest of £1 for a year)

$$\mathrm{a}^{(p)}=\frac{1-r\mathrm{A}^{(p)}}{1+r}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (1)$$

$$\therefore \; 1-\mathrm{a}^{(p)}=\frac{r}{1+r}(1+\mathrm{A}^{(p)}), \quad . \; . \; . \; . \; . \; . \; . \qquad (2)$$

and

$$1+\mathrm{A}^{(p)}=\left(1+\frac{1}{r}\right)(1-\mathrm{a}^{(p)}). \; . \; . \; . \; . \; . \; . \qquad (3)$$

We have evidently $\mathrm{a}_{,}=\mathrm{a}'$; also when A'' shall be nominated the value of £1 payable at his death will be a''; but the present value of £1 payable when A'' shall be nominated, that is, at the end of the year in which A' shall die, is $\mathrm{a}_{,}$; hence the present value of £1 payable at the death of A'' is

$$\mathrm{a}\times\mathrm{a}''=\mathrm{a}'\mathrm{a}'';$$
$$\therefore \; \mathrm{a}_2=\mathrm{a}'\mathrm{a}''.$$

And generally when the life $\mathrm{A}^{(p)}$ shall be nominated, the then value of £1 payable at his death will be $\mathrm{a}^{(p)}$; but the present value of £1 payable at the death of $\mathrm{A}^{(p-1)}$, that is, at the nomination of $\mathrm{A}_{(p)}$, is a_{p-1},

$$\therefore \; \mathrm{a}_p=\mathrm{a}_{p-1}.\mathrm{a}^{(p)}.$$

Hence taking $p=1, 2, 3 \ldots . n$ in succession, we have

$$\begin{aligned} &\mathrm{a}_1=\mathrm{a}' \\ &\mathrm{a}_2=\mathrm{a}_1\mathrm{a}'' \\ &\mathrm{a}_3=\mathrm{a}_2\mathrm{a}''' \\ &\vdots \\ &\mathrm{a}_n=\mathrm{a}_{n-1}.\mathrm{a}^{(n)}. \end{aligned}$$

Multiply these equations together and cancel the common factors

$$\mathrm{a}_n=\mathrm{a}'\mathrm{a}''\mathrm{a}''' \ldots . \mathrm{a}^{(n)} \; . \; . \; . \; . \; . \; . \; . \; . \qquad (4)$$

Hence the present value of an assurance payable on the failure of the last of any number of successive lives is very readily computed, provided we have a table of the present values of assurances on single lives; and such tables are given in David Jones's work on *Annuities and Reversionary Payments* (Tables IX. and XXII.), according to both the Northampton and Carlisle tables of mortality. If however only a table of annuities be at hand, it will be better to modify (4) as follows:—

By (1) we have $\mathrm{a}^{(p)}=v(1-r\mathrm{A}^{(p)})$, where $v=(1+r)^{-1}$, the present value of £1, due in a year, hence (4) becomes

$$\mathrm{a}_n=v^n(1-r\mathrm{A}').(1-r\mathrm{A}'')\ldots.(1-r\mathrm{A}^{(n)}). \quad . \quad . \quad . \quad . \qquad (5)$$

Problem II.—To determine (a_p) the present value of an annuity of £1 on the pth life mentioned in Problem I.

At the nomination of $\mathrm{A}^{(p)}$ £1 will be due, and the annuity on his life will be worth $\mathrm{A}^{(p)}$; hence the life will then be worth $1+\mathrm{A}^{(p)}$; but £1 due at the nomination of $\mathrm{A}^{(p)}$ is now worth a_{p-1}; hence the present value of the annuity on the pth life is

$$\mathrm{a}_{p-1}.(1+\mathrm{A}^{(p)}),$$

$$\therefore, (4), \ a_p=\mathrm{a}'\mathrm{a}''\ldots.\mathrm{a}^{(p-1)}.(1+\mathrm{A}^{(p)}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and this is the present value of £1 per annum on the pth life.

The preceding (6) is the form that will generally be found best adapted to computation; but a_p may also be expressed in terms of a only, or of A only, as follows (see (1) and (3)):—

$$a_p=\left(1+\frac{1}{r}\right)\mathrm{a}'\mathrm{a}''\ldots.\mathrm{a}^{(p-1)}(1-\mathrm{a}^{(p)}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

and

$$a_p=v^{p-1}(1-r\mathrm{A}')(1-r\mathrm{A}'')\ldots.(1-r\mathrm{A}^{(p-1)})(1+\mathrm{A}^{(p)}) \quad . \qquad (8)$$

It may be observed too that (7) is little, if any, inferior to (6) in point of easy application, as it requires a reference to one table only, while (6) requires a reference to two.

Note.—Unless the annuity be *due*—that is, unless a payment is to be made immediately—none of the preceding formulas will give a_1, the value of the annuity on the first life, correctly. If we suppose, as usual, that the first payment will be made in a year, the true value of a_1 will be

$$a_1=\mathrm{A}'=\left(1+\frac{1}{r}\right)(1-\mathrm{a}')-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and not

$$a_1=1+\mathrm{A}'=\left(1+\frac{1}{r}\right)(1-\mathrm{a}').$$

Problem III.—To determine (A_n), the present value of an annuity of £1 on the n successive lives A', A'' .... $\mathrm{A}^{(n)}$.

We evidently have

$$\mathrm{A}_n=a_1+a_2+a_3\ldots.+a_n.$$

But (7) and (9)—

$$a_1=\left(1+\frac{1}{r}\right)(1-\mathrm{a}')-1$$

$$a_2=\left(1+\frac{1}{r}\right)(\mathrm{a}'-\mathrm{a}'\mathrm{a}'')$$

$$a_3=\left(1+\frac{1}{r}\right)(\text{a}'\text{a}''-\text{a}'\text{a}''\text{a}''')$$

$$\vdots$$

$$a_n=\left(1+\frac{1}{r}\right)(\text{a}'\text{a}''\ldots\text{a}^{(n-1)}-\text{a}'\text{a}''\ldots\text{a}^{(n)}).$$

Hence by addition we have

$$\text{A}_n=\left(1+\frac{1}{r}\right)(1-\text{a}'\text{a}''\ldots\text{a}^{(n)})-1 \quad \ldots \quad (10)$$

It appears from (4) that (10) is equivalent to

$$\text{A}_n=\left(1+\frac{1}{r}\right)(1-\text{a}_n)-1;$$

and, in fact, this might have been deduced immediately from (3), for the formula (1) which expresses the relation between an annuity and the corresponding assurance, is true of any status*, providing the status be such that the assurance on it *must* be paid some time or other.

PROBLEM IV.—A copyhold estate is held on a certain number of lives A, B, C .... and each life is renewable at the end of the year in which it may fail by paying a fine of f pounds. Required the present value of all the fines.

Let A′, A″ .... be the lives that succeed A; B′, B″ .... those that succeed B, &c.; also let a, a′ a″ .... be the values of assurances on the lives A, A′, A″ ....; b, b′, b″.... those on B, B′, B″ ...., &c.

By (4) it appears that the fines (each equal to f) payable at the deaths of A, A′, A″ .... are now worth fa, $f$aa′, $f$aa′a″ .... respectively; and similar expressions being true of the other successive lives, the present value of all the fines is

$$f\times\left\{\begin{array}{l}\text{a}+\text{aa}'+\text{aa}'\text{a}''+\ldots\ldots\\+\text{b}+\text{bb}'+\text{bb}'\text{b}''+\ldots\ldots\\+\text{c}+\text{cc}'+\text{cc}'\text{c}''+\ldots\ldots\\+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \quad \ldots\ldots \quad (11)$$

If the lives, at the time of the nomination, be all of the same age (which must be assumed in practice), then, P denoting the value at the time of nomination of an annuity of £1 on each renewal life, and p the assurance on the same, we shall have

$$\text{a}'=\text{a}''=\ldots=\text{b}'=\text{b}''=\ldots\ldots=\text{c}'=\text{c}''=\ldots\ldots\ldots=\text{p},$$

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* "By the *status* of an annuity, I mean the state or condition of things during the continuance of which the annuity is to be paid."—De Morgan's *Essay on Probabilities*, p. 190.

and each horizontal row in (11) now constitutes a geometrical progression of which the common ratio is p; and the sum of the whole is evidently

$$f.\left\{\frac{\mathrm{a}}{1-\mathrm{p}}+\frac{\mathrm{b}}{1-\mathrm{p}}+\frac{\mathrm{c}}{1-\mathrm{p}}+\ldots.\right\}$$

Hence

$$f.\frac{\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots\ldots..}{1-\mathrm{p}}, \quad \ldots\ldots. \quad (12)$$

is the present value of all the fines.

The preceding expression (12) may also be exhibited in terms of the annuities on the lives instead of the assurances on the same. For

$$(1),\ \mathrm{a}=\frac{1-rA}{1+r},\ \mathrm{b}=\frac{1-rB}{1+r},\ \ldots,\ \text{and, (2)},\ 1-\mathrm{p}=\frac{r}{1+r}(1+P),$$

hence (12) becomes

$$f.\frac{n\frac{1}{r}-(A+B+C+\ldots.)}{1+P}, \quad \ldots\ldots \quad (13)$$

where n denotes the number of lives A, B, C ... on which the estate is held, and it may be observed that $\frac{1}{r}$ is a perpetuity of £1.

The formulas (6), (10), (11) and (12), agree with those given by Mr. Milne, but they are here expressed in a way better adapted to computation. That these coincide with Mr. Milne's expressions, will readily appear, if, instead of A′ ..., we introduce annuities certain of equal values; thus let $A^{(p)}$ be equivalent to an annuity certain for t_p years, so that

$$A^{(p)}=\frac{1-v^{t_p}}{r},$$

$$\therefore\ 1+A^{(p)}=\left(1+\frac{1}{r}\right)(1-v^{t_p+1}),\ \text{and, (2)},\ \mathrm{a}^{(p)}=v^{t_p+1},$$

hence (10) becomes

$$A_n=\left(1+\frac{1}{r}\right)(1-v^{\sigma+1})-1,\ (\sigma=n-1+t_1+t_2\ldots.+t_n),$$

or

$$A_n=\frac{1-v^{\sigma}}{r},$$

which is Mr. Milne's formula; and similarly, it may be shown that (6) and (11) coincide with the expressions given in Milne.

Also if P be equivalent to an annuity certain of t years, so that $P=\frac{1-v^t}{r}$, and therefore $p=v^{t+1}$, then (12) becomes

$$f\frac{a+b+c+\dots\dots}{1-v^{t+1}}\,;$$

which is Mr. Milne's expression.

The formula (13) is not in Milne, but is given by Professor De Morgan in his *Essay on Probabilities* (*Cab. Cyclop.*), Appendix the Second. It is very well adapted to computation, though not I think quite so well as (12), if Jones's tables previously alluded to be used. The difference between the two formulas in point of practical application is, however, very trifling.

Eighth Census of the United States, in 1860. *By* SAMUEL BROWN, *Esq.*, V.P.S.S.

IN the course of last year a very important document was published by the American Government, under the direction of the Secretary of the Interior, entitled "Statistics of the United States (including Mortality, Property, &c.) in 1860; compiled from the original Returns, and being the final Exhibit of the Eighth Census." It contains an introduction by J. M. Edmunds, Commissioner of General Land Office, in charge of census, and a still fuller introduction to the mortality statistics for the year ending June 1st, 1860, by Dr. Edward Jarvis, a very able writer, who was a delegate to the Statistical Congress, when it was held in London, and is a Corresponding Member of the Statistical Society of London.

The former introduction gives a brief notice of the census as taken in different countries (commencing with the enumeration of the people under the Mosaic Dispensation) to the first census of the United States, in 1790. Since that time there has been one every 10 years, and a short sketch is given of the legal enactments by which they were carried out. The statute of 23rd May, 1850, by which the seventh census was ordered, also made provision for the eighth and any subsequent census, which was supplemented in 1860 and 1862 by some Acts which required, amongst other things, the Secretary of War to be furnished with such war statistics as might be needed, and gave a credit of two millions of dollars =£400,000 for the necessary expenses, including the costs of printing and binding.

It does not appear clear from this Report whether the census was to be taken for any single day, or, better still, for any single night; but merely that the schedules, after being sent out to the marshals, were to be returned by some specified date. If so, the results are liable to all the errors arising from many persons and facts being counted twice over, and some not being included at all. The actual population might therefore be very different, except under the extreme improbability that the changes of population on different days were exactly in the same proportion at all ages.

The forms of schedules comprise classifications for the produce of agriculture, of industry, social statistics, public libraries, periodicals, newspapers, pauperism, crime, cost of labour, religious worship, and mortality statistics.

Before giving a brief summary of the last, which are likely to be more especially interesting to the readers of this *Journal,* it may be well to quote the remarkable figures which form the totals of some of the preceding heads of classification.

Population.—At the close of the sixteenth century, the regions comprised within the limits of the United States were described as a wilderness, untrodden by civilized man.

A century after, in the year 1700, the inhabitants of the Anglo-American colonies are stated to be	262,000
In 1749–50, the estimated population was .	1,000,000
In 1775, estimated by Congress in view of the approaching rupture	2,389,000
In 1790, at the first census according to law .	3,930,000
In 1860, by the eighth census, computed to be	31,443,000

Agriculture.—Under the Act of 1798, the number of acres assessed for direct taxes, independent of lands belonging to the United States, or exempted from taxation, and exclusive of Louisiana, amounted to 163,747,000 acres, and valued at 479,293,000 dollars =£95,859,000. In 1809, the *improved* lands in the United States, including pastures, embraced 63,570,000 acres. In 1860, the lands are thus stated—

	Improved, in Farms. Acres.	Unimproved. Acres.	Cash Value. Dollars.	
			Dollars.	£
In the States .	162,650,000	241,944,000	6,631,520,000 =	1,326,304,000
In the Territories	461,000	2,158,000	13,525,000 =	2,705,000
Total . .	163,111,000	244,102,000	6,645,045,000 =	1,329,009,000

In 1850, the number of acres of improved lands was 113,033,000, and the cash value of farms, 3,271,575,000 dollars =£654,315,000, showing an increase in the 10 years of 44 per cent. in improved lands, and more than 50 per cent. in the values.

Manufactures.—As to the annual value of manufactured products, the estimates at all the periods must be somewhat more than vague. Up to 1790, they were mostly confined to those of the household class, although the manufactures of iron and its ultimate products, coarse woollens, and papers, were becoming greatly extended by means of joint stock and incorporated Companies. From 1810 to 1860, the product is stated in the volume for manufactures to have increased from 198,613,000 dollars = £39,723,000 in the former year to the enormous sum of 2,000,000,000 dollars = £400,000,000 in 1860.

The fourth volume of the census for 1860 treats of

Mortality.—This subject is more fully discussed in the Report of Dr. Jarvis, who has taken much pains to classify the deaths, not merely as to age and sex, and whether of the white or coloured population, but to give the causes of death and the diseases of the two races according to the subdivisions adopted in the English Registration Office and other continental returns. There is great reason, however, to doubt the accuracy of the facts collected by the census officers as to the deaths in the course of the preceding year, except where they are confirmed by a previously-existing system of registration. The general result is stated to be that, in 1850, with a population of 23 millions, 323,000 persons died, or 14 per 1,000; whilst in 1860, with a population of 31½ millions, only 394,000 deaths occurred, or 12¼ per 1,000; both proportions so low as to suggest the idea either that some serious error has been committed, or that some fuller explanations are required as to the causes. Yet the conclusion in the report is that with such a rapid increase of rate of population over rate of mortality, "the beginning of the year 1900 will find within the present limits of this Republic 107 millions of inhabitants."

Banks.—On the 1st January, 1811, the whole number of banks in the United States was 88, their aggregate capital 22,700,000 dollars = £4,500,000, and of specie 9,600,000 dollars = £1,920,000. In 1850 there were 872 banks, capital 227 millions of dollars = £45,400,000; and in 1860, 1,562, with a capital of 421 millions of dollars = £84,200,000.

On the 1st January, 1866, the amount estimated was 380 millions of dollars of National Bank notes = £76,000,000, 80 millions of dollars = £16,000,000 from State Banks, 129 millions of dollars = £25,800,000, supposed to have been issued since 1st October, 1865, to National Banks; whilst the gold and silver products from mines, for the year ending 30th June, 1865, were 100 millions of dollars = £20,000,000, and the receipts into the Treasury for that year amounted to 929½ millions of dollars = £185,900,000.

Insurance.—The earliest Insurance Offices in America are supposed to be one established at Boston in 1724, and one in Philadelphia in 1756.

In 1860 the number of Insurance Companies in the United States was 294; their capitals and assets 82,170,000 dollars = £16,434,000; at risk, 2,605½ millions of dollars = £521,100,000; losses in marine and fire branches in 1860, 50,596,000 dollars = £10,119,000.

For life assurance there were 47 Offices, embracing 60,000 lives, with assurances for 180 millions of dollars = £36,000,000; the annual premiums being stated to be 7 millions of dollars = £1,400,000, which, if correct, would be very much in excess of the average rate in this country, unless there is comprised in that item the interest from investments also.

Railroads and canals.—In 1860 the commercial railroads are reported to have a total length of 30,794 miles, and the cost of construction to have been 1,151½ millions of dollars = £230,300,000; whilst the city passenger railways were equal to 403 miles, costing 14,863,000 dollars = £2,972,600.

The first canal completed in the United States was the Middlesex, between Boston Harbour and Concord River, 27 miles in length, which was constructed, at a cost of 550,000 dollars = £110,000, by a Company incorporated in 1789. In 1860 there were 118 canals, having a total length of 5,462 miles; the cost of construction in 68 of which was 147,400,000 dollars = £29,475,000, the cost of the remainder not being reported.

Value of real and personal property.—In 1789 the total property valuation for taxation consisted of 163,750,000 acres and 277,000 houses, together valued at 620 millions of dollars = £124,000,000. In 1850 the estimate of value rose to 7,136 millions of dollars = £1,427,160,000; whilst in 1860, according to the marshals' returns, the value had risen to 16,160 millions of dollars = £3,232,000,000. The aggregate of the individual returns of real and personal private property in the country is still higher, making the estimate 19,089 millions of dollars = £3,817,800,000.

Educational establishments.—The total number of educational establishments in 1860, including collegiate, academies, and public schools was 113,000, in which were employed 149,000 teachers, giving instruction to 5,418,000 persons, and possessing a total annual income of 33,990,000 dollars = £6,798,000. The number of libraries returned in 1860 was 27,730, with 13,316,000 volumes.

Religious establishments.—The returns in 1860 gave 54,000 churches, the value of their property being 172 millions of dollars = £34,400,000; being capable of affording accommodation to 18,975,000 persons, averaging one church to every 584 individuals in the whole population.

Manufactures, postal transit, and the press.—It is stated that in 1860 there were amongst the large towns 102, which contained an aggregate population of 4,764,000, with a manufacturing capital of 417 millions of dollars = £83,400,000, employing upwards of 557,000 persons; and the value of the manufactured produce realized the sum of 875 millions of dollars = £175,000,000.

The postal service is given to a more recent date, the beginning of the fiscal year 1865, when it embraced 6,012 routes, of an aggregate length of 142,000 miles, at a total cost of 6,803,000 dollars = £1,360,000; the aggregate transportation being equal to 57,993,000 miles.

The power of the public press is well known in America, and it seems to be growing with increased rapidity, for whereas in 1850 there were reported 2,526 newspapers of all kinds, with a circulation of 426,409,000, in 1860 they amounted to 4,051 newspapers, with an annual circulation of 928 millions of copies. The annual receipts of a single leading paper are stated to be more than one million of dollars, or £200,000. The first journal published in the Anglo-American Colonies was the *Boston Newsletter*, in 1704; and at the opening of the revolution only 40 newspapers were in existence.

Army and Navy.—The summary of the army and navy is not given according to the census year—a year of peace, but a comparison is made at different *war periods*, in 1775, 1812, and 1865. At the last date, on the 10th March, the navy is stated to consist of 684 ships of war, with 4,477 guns, and an aggregate tonnage of 519,000 tons. At the end of 1864 the naval service is said to have comprised 6,000 officers and 45,000 men.

The number of men raised for the service of the Union armies during the late war is computed at nearly 2,688,000, though no doubt this comprises the numbers actually voted, and probably large deductions would have to be made for such as did not appear in the field. From the actual number returned must also be taken off many who enlisted and received bounty money more than once. Including the Confederate armies, 4,000,000 of men are computed to have engaged in the civil war, or nearly one-seventh of the whole population.

In all the above quoted large numbers and values the figures have been expressed in the nearest thousands, as quite sufficiently accurate for the picture presented by this census of the remarkable progress and material prosperity of the great American nation. These results are, of course, only the summaries of numerous tables. The details are given with much minuteness, and show the vast variety of information as to the condition of a people which a census, if accurately taken, is capable of affording. The Report by Dr. Jarvis, on the mortality statistics, involves some points of such great and curious interest, especially as regards the white and coloured populations, that it may be worth analyzing separately on a future occasion.

A Budget of Paradoxes. By PROFESSOR DE MORGAN.

(Continued from page 188.)

No. XVIII. 1847—1849.

Aerial Navigation; containing a description of a proposed flying machine, on a new principle. By Dædalus Britannicus. London, 1847, 8vo.

In 1842-43 a Mr. Henson had proposed what he called an aeronaut steam-engine, and a Bill was brought in to incorporate an "Aerial Transit Company." The present plan is altogether different, the moving power being the explosion of mixed hydrogen and air. Nothing came of it—not even a Bill. What the final destiny of the balloon may be no one knows: it may reasonably be suspected that difficulties will at last be overcome. Darwin, in his *Botanic Garden* (1781), has the following prophecy:—

"Soon shall thy arm, unconquered Steam! afar
Drag the slow barge, or drive the rapid car;
Or, on wide-waving wings expanded, bear
The flying chariot through the fields of air."

Darwin's contemporaries, no doubt, smiled pity on the poor man. It is worth note that the two true prophecies have been fulfilled in a sense different from that of the predictions. Darwin was thinking of the suggestion of Jonathan Hulls, when he spoke of dragging the slow barge: it is only very recently that the steam-tug has been employed on the canals. The car was to be driven, not drawn, and on the common roads. Perhaps, the flying chariot will be something of a character which we cannot imagine, even with the two prophecies and their fulfilments to help us.

A book for the public. New discovery. The causes of the circulation of the blood; and the true nature of the planetary system. London, 1848, 8vo.

Light is the sustainer of motion both in the earth and in the blood. The natural standard, the pulse of a person in health, four beats to one respiration, gives the natural second, which is the measure of the earth's progress in its daily revolution. The Greek fable of the Titans is an elaborate exposition of the atomic theory: but any attempt to convince learned classics would only meet their derision; so much does long-fostered prejudice stand in the way of truth. The author complains bitterly that men of science will not attend to him and others like him: he observes, that "in the time occupied in declining, a man of science might test the merits." This is, alas! too true; so well do applicants of this kind know how to stick on. But every rule has its exception: I have heard of one. The late Lord Spencer—the Lord Althorp of the House of Commons—told me that a speculator once got access to him at the Home Office, and was proceeding to unfold his way of serving the public. "I do not understand these things," said Lord Althorp, "but I happen to have —— (naming an eminent engineer) upstairs; suppose you talk to him on the subject." The discoverer went up, and in half-an-hour returned, and said, "I am very much obliged to your Lordship for introducing me to Mr. ——; he has convinced me that I am quite wrong." I supposed, when I heard the story—but it would not have been seemly to say it—that Lord A. perspired candour and sense, which infected those who came within reach: he would have done so, if anybody.

A method to trisect a series of angles having relation to each other; also another to trisect any given angle. By James Sabben. 1848. (Two quarto pages.)

"The consequence of years of intense thought": very likely, and very sad.

1848. The following was sent to me in manuscript. I give the whole of it:—

Quadrature of the Circle.—A quadrant is a curvilinear angle traversing round and at an equal distance from a given point, called a centre, no two points in the curve being at the same angle, but irreptitiously graduating from 90 to 60. It is therefore a mean angle of 90 and 60, which is 75, because it is more than 60, and less than 90, approximately from 60 to 90, and from 90 to 60, with equal generation in each irreptitious approximation, therefore meeting in 75, and which is the mean angle of the quadrant.

Or, suppose a line drawn from a given point at 90, and from the same point a line at 60. Let each of these lines revolve on this point toward each other at an equal ratio. They will become one line at 75, and bisect the curve, which is one-sixth of the entire circle. The result, taking 16 as a diameter, gives an area of 201·072400, and a circumference of 50·2681.

The original conception, its natural harmony, and the result, to my own mind is a demonstrative truth, which I presume it right to make known, though perhaps at the hazard of unpleasant if not uncourteous remarks.

I have added punctuation: the handwriting and spelling are those of an educated person; the word *irreptitious* is indubitable. The whole is a natural curiosity.

The quadrature and exact area of the circle demonstrated. By Wm. Peters. 8vo. *n. d.* (circa 1848).

Suggestions as to the necessity for a revolution in philosophy; and prospectus for the establishment of a new quarterly, to be called the *Physical Philosopher and Heterodox Review.* By Q. E. D. 8vo. 1848.

These works are by one author, who also published, as appears by advertisement,

> "Newton rescued from the precipitancy of his followers through a century and a half," and "Dangers along a coast by correcting (as it is called) a ship's reckoning by bearings of the land at night fall, or in a fog, nearly out of print. Subscriptions are requested for a new edition."

The area of a circle is made four-fifths of the circumscribed square: proved on an assumption which it is purposed to explain in a longer essay. The author, as Q. E. D., was in controversy with the *Athenæum* journal, and criticized a correspondent, D., who wrote against a certain class of discoverers. He believed the common theories of hydrostatics to be wrong, and one of his questions was—

> "Have you ever taken into account anent gravity and gravitation the fact that a five grain cube of cork will of itself half sink in the water, whilst it will take 20 grains of brass, which will sink of itself, to pull under the other half? Fit this if you can, friend D., to your notions of gravity and specific gravity, as applied to the construction of a universal law of gravitation."

This the *Athenæum* published—but without some Italics, for which the editor was sharply reproved, as a sufficient specimen of the *quod erat* D. *monstrandum:* on which the author remarks—"D,—Wherefore the e caret? is it D apostrophe? D', D'M, D'Mo, D'Monstrandum; we cannot find the *wit* of it." This I conjecture to contain an allusion to the name of the supposed author; but whether De Mocritus, De Mosthenes, or De Moivre was intended, I am not willing to decide.

> The scriptural Calendar and Chronological Reformer, for the statute year 1849. Including a review of recent publications on the Sabbath question. London, 1849, 12mo.

This is the almanac of a sect of Christians who keep the Jewish Sabbath, having a chapel at Mill Yard, Goodman's Fields. They wrote controversial works, and perhaps do so still; but I never chanced to see one.

> Geometry *versus* Algebra; or the trisection of an angle geometrically solved. By W. Upton, B.A. Bath (circa 1849). 8vo.

The author published two tracts under this title, containing different alleged proofs: but neither gives any notice of the change. Both contain the same preface, complaining of the British Association for refusing to examine the production. I suppose that the author, finding his first proof wrong, invented the second, of which the Association never had the offer; and feeling sure that they would have equally refused to examine the second, thought it justifiable to present that second as the one they had refused. Mr. Upton has discovered that the common way of finding the circumference is wrong, would set it right if he had leisure, and, in the meantime, has solved the problem of the duplication of the cube.

The trisector of an angle, if he demand attention from any mathematician, is bound to produce, from his construction, an expression for the sine or cosine of the third part of any angle, in terms of the sine or cosine of the angle itself, obtained by help of no higher than the square root. The mathematician knows that such a thing cannot be; but the trisector virtually says it can be, and is bound to produce it, to save time. This is the misfortune of most of the solvers of the celebrated problems, that they have not knowledge enough to present those consequences of their results by which they can be easily judged. Sometimes they have the knowledge, and quibble out of the use of it. In many cases a person makes an honest beginning and presents what he is sure is a solution. By conference with others he at last feels uneasy, fears the light, and puts self-love in the way of it. Dishonesty sometimes follows. The speculators are, as a class, very apt to imagine that the mathematicians are in fraudulent confederacy against them: I ought rather to say that each one of them consents to the mode in which the rest are treated, and fancies conspiracy against himself. The mania of conspiracy is a very curious subject. I do not mean these remarks to apply to the author before me.

One of Mr. Upton's trisections, if true, would prove the truth of the following equation :—

$$3\cos\frac{\theta}{3}=1+\surd(4-\sin^2\theta),$$

which is certainly false.

In 1852 I examined a terrific construction, at the request of the late Dr. Wallich, who was anxious to persuade a poor countryman of his that trisection of the angle was waste of time. One of the principles was, that "magnitude and direction determine each other." The construction was equivalent to the assertion that, θ being any angle, the cosine of its third part is

$$\sin 3\theta.\cos\frac{5\theta}{2}+\sin^2\theta\,\sin\frac{5\theta}{2}$$

divided by the square root of

$$\sin^2 3\theta\,\cos^2\frac{5\theta}{2}+\sin^4\theta+\sin 3\theta.\sin 5\theta.\sin^2\theta.$$

This is from my rough notes, and I believe it is correct. It is so nearly true, unless the angle be very obtuse, that common drawing, applied to the construction, will not detect the error. There are many formulæ of this kind: and I have several times found a speculator who has discovered the corresponding construction, has seen the approximate success of his drawing—often as great as absolute truth could give in graphical practice—and has then set about his demonstration, in which he always succeeds to his own content.

There is a trisection of which I have lost both cutting and reference: I think it is in the *United Service Journal.* I could not detect any error in it, though certain there must be one. At least I discovered that two parts of the diagram were incompatible unless a certain point lay in line with two others, by which the angle to be trisected—and which was trisected—was bound to be either 0° or 180°.

'No. XIX. 1849—1850.

Notes on the Kinematic effects of Revolution and Rotation, with reference to the Motions of the Moon and of the earth. By Henry Perigal, Jun., Esq. London, 1846–1849, 8vo.

On the misuse of technical terms. Ambiguity of the terms *Rotation* and *Revolution*, owing to the double meaning improperly attributed to each of the words. (No date nor place, but by Mr. Perigal, I have no doubt, and containing letters of 1849 and 1850.)

The moon controversy. Facts *v.* Definitions. By H. P., Jun. London, 1856, 8vo. (pp. 4.)

Mr. Henry Perigal helped me twenty years ago with the diagrams, direct from the lathe to the wood, for the article "Trochoidal Curves," in the *Penny Cyclopædia*: these cuts add very greatly to the value of the article, which, indeed, could not have been made intelligible without them. He has had many years' experience, as an amateur turner, in combination of double and triple circular motions, and has published valuable diagrams in profusion. A person to whom the double circular motion is familiar in the lathe naturally looks upon one circle moving upon another as in *simple* motion, if the second circle be fixed to the revolving radius, so that one and the same point of the moving circle travels upon the fixed circle. Mr. Perigal commenced his attack upon the moon for moving about her axis, in the first of the tracts above, ten years before Mr. Jellinger Symons; but he did not think it necessary to make it a subject for the *Times* newspaper. His familiarity with combined motions enabled him to handle his arguments much better than Mr. J. Symons could do: in fact, he is the clearest assailant of the lot which turned out with Mr. J. Symons. But he is as wrong as the rest. The assault is now, I suppose, abandoned, until it becomes epidemic again. This it will do: it is one of those fallacies which are very tempting. There was a dispute on the subject in 1748, between James Ferguson and an anonymous opponent; and I think there have been others.

A poet appeared in the field (July 19, 1863) who calls himself Cyclops, and writes four octavo pages. He makes a distinction between *rotation* and *revolution*; and his doctrines and phrases are so like those of Mr. Perigal, that he is a follower, at least. One of his arguments has so often been used that it is worth while to cite it:—

> Would Mathematicals—forsooth—
> If true, have failed to prove its truth?
> Would not they—if they could—submit
> Some overwhelming proofs of it?
> But still it totters *proofless!* Hence
> There's strong presumptive evidence
> None do—or can—such proof propound
> Because *the dogma is unsound.*
> For, were there means of doing so,
> They would have proved it long ago.

This is only one of the alternatives. Proof requires a person who can give and a person who can receive. I feel inspired to add the following:—

A blind man said, As to the Sun,
I'll take my Bible oath there's none;
For if there had been one to show
They would have shown it long ago.
How came he such a goose to be?
Did he not know he could'nt see?
Not he!

The solar system truly solved; demonstrating, by the perfect harmony of the planets, founded on the four universal laws, the Sun to be an electrical space; and a source of every natural production displayed throughout the solar system. By James Hopkins. London, 1849, 8vo.

The author says:—

"I am satisfied that I have given the true *laws* constituting the *Sun* to be *space;* and I call upon those disposed to maintain the contrary, to give true *laws* showing him to be a body: until such can be satisfactorily established, I have an undoubted claim to the credit of my theory,—That the Sun is an *Electric Space*, fed and governed by the planets, which have the property of attracting heat from it; and the means of supplying the necessary *pabulum* by their degenerated air driven off towards the central space—the wonderful alembic in which it becomes transmuted to the revivifying necessities of continuous action; and the central space or Sun being perfectly electric, has the counter property of repulsing the bodies that attract it. How wonderful a conception! How beautiful, how magnificent an arrangement!

"O Centre! O Space! O Electric Space!"

1849. *Joseph Ady* is entitled to a place in this list of discoverers: his great fault, like that of some others, lay in pushing his method too far. He began by detecting unclaimed dividends, and disclosing them to their right owners, exacting his fee before he made his communication. He then generalized into trying to get fees from all of the *name* belonging to a dividend; and he gave mysterious hints of dangers impending. For instance, he would write to a clergyman that a legal penalty was hanging over him; and when the alarmed divine forwarded the sum required for disclosure, he was favoured with an extract from some old statute or canon, never repealed, forbidding a clergyman to be a member of a Corporation, and was reminded that he had insured his life in the —— Office, which had a royal charter. He was facetious, was Joseph: he described himself in his circulars as "personally known to Sir Peter Laurie and all other aldermen"; which was nearly true, as he had been before most of them on charges of false pretence;

but I believe he was nearly always within the law. Sir James Duke, when Lord Mayor, having particularly displeased him by a decision, his circulars of 1849 contain the following:—

> "Should you have cause to complain of any party, Sir J. Duke has contrived a new law of evidence, viz., write to him, he will consider your letter sufficient proof, and make the parties complained of pay without judge or jury, and will frank you from every expense."

I strongly suspect that Joseph Ady believed in himself.

He sometimes issued a second warning, of a Sibylline character:—

> "Should you find cause to complain of anybody, my voluntary referee, the Rt. Hon. Sir Peter Laurie, Kt., perpetual Deputy Lord Mayor, will see justice done you without any charge whatever: he and his toady, — —— ——. The accursed of Moses can hang any man: thus, by catching him alone and swearing Naboth spake evil against God and the King. Therefore (!) I admit no strangers to a personal conference without a prepayment of 20*s.* each. Had you attended to my former notice you would have received twice as much: neglect this and you will lose all."

Zadkiel's Almanac for 1849. Nineteenth number.

Raphael's Prophetic Almanac for 1849. Twenty-ninth number.

Reasons for belief in judicial astrology, and remarks on the dangerous character of popish priestcraft. London, 1849, 12mo.

Astronomy in a nutshell: or the leading problems of the solar system solved by simple proportion only, on the theory of magnetic attraction. By Lieut. Morrison, R.N. London (*s. a.*) 12mo.

Lieut. Morrison is Zadkiel Tao Sze, and declares himself in real earnest an astrologer. There are a great many books on astrology, but I have not felt interest enough to preserve many of them which have come in my way. If anything ever had a fair trial, it was astrology. The idea itself is natural enough. A human being, set down on this earth, without any tradition, would probably suspect that the heavenly bodies had something to do with the guidance of affairs. I think that any one who tries will ascertain that the planets do not prophesy: but if he should find to the contrary, he will of course go on asking. A great many persons class together belief in astrology and belief in apparitions: the two things differ in precisely the way in which a science of observation differs from a science of experiment. Many make the mistake which M. le Marquis made when he came too late, and hoped M. Cassini would do the eclipse over again for his ladies. The apparition chooses its own time, and comes as seldom or as often as it pleases, be it departed spirit, nervous derangement, or imposition. Consequently it can only be observed, and not experimented upon. But the heavens, if astrology be true, are prophesying away day and night all the year round, and about every body. Experiments can be made, then, except only on rare phenomena, such as eclipses: anybody may choose his time and his question. This is the great difference: and experiments were made, century after century. If astrology had been true, it must have lasted in an ever-improving state. If it be true, it is a truth, and a useful truth, which had experience and prejudice both in its favour, and yet lost ground as soon as astronomy, its working tool, began to improve.

1850. A letter in the handwriting of an educated man, dated from a street in which it must be taken that educated persons live, is addressed to the Secretary of the Astronomical Society about a matter on which the writer says "his professional pursuit will enable him to give a satisfactory reply" In a question before a court of law it is sworn on one side that the moon was shining at a certain hour of a certain night on a certain spot in London; on the other side it is affirmed that she was clouded. The Secretary is requested to decide. This is curious, as the question is not astrological. Persons still send to Greenwich, now and then, to have their fortunes told. In one case, not very many years ago, a young gentleman begged to know who his wife was to be, and what fee he was to remit.

Sometimes the astronomer turns conjuror for fun, and his prophecies are fulfilled. It is related of Flamsteed that an old woman came to know the whereabouts of a bundle of linen which had strayed. Flamsteed drew a circle, put a square into it, and gravely pointed out a ditch, near her cottage, in which he said it would be found. He meant to have given the woman a little good advice when she came back: but she came back in great delight, with the bundle in her hand, found in the very place. The late Baron Zach received a letter from Pons, a successful finder of comets, complaining that for a certain period he had found no comets, though he had searched diligently. Zach, a man of much sly humour, told him that no spots had been seen on the sun for about the same time—which was true,—and assured him that when the spots came back, the comets would come with them. Some time after he got a letter from Pons, who informed him with great satisfaction that he was quite right, that very large spots had appeared on the sun, and that he had found a fine comet shortly after. I do not vouch for the first story, but I have the second in Zach's handwriting. It would mend the joke exceedingly if some day a real relation should be established between comets and solar spots: of late years good reason has been shown for advancing a connexion between these spots and the earth's magnetism. If the two things had been put to Zach, he would probably have chosen the comets. Here is a hint for a paradox: the solar spots are the dead comets, which have parted with their light and heat to feed the sun, as was once suggested. I should not wonder if I were too late, and the thing had been actually maintained. My list does not contain the twentieth part of the possible whole.

April 1850, found in the letter-box, three loose leaves, well printed, and over punctuated, being

> Chapter VI. Brethren, lo I come, holding forth the world of life, for so I am commanded.... Chapter VII. Hear my prayer, O generations! and walk by the way, to drink the waters of the river.... Chapter VIII. Hearken o earth, earth, earth, and the kings of the earth, and their armies....

A very large collection might be made of such apostolic writings. They go on well enough in a misty—meant for mystical—imitation of St. Paul or the prophets, until at last some prodigious want of keeping shows the education of the writer. For example, after half a page which might pass for Irving's preaching—though a person to whom it was presented as such would say that most likely the head and tail would make something more like head and tail of it—we are astounded by a declaration from the *Holy Spirit*, speaking of himself, that he is "not ashamed of the Gospel of Christ." It would be long before we should find in *educated* rhapsody—of which there are specimens enough—such a thing as a person of the Trinity taking merit for moral courage enough to stand where St. Peter fell. The following declaration comes next —"I will judge between cattle and cattle, that use their tongues."

> The figure of the earth. By J. L. Murphy, of Birmingham. (London and Birmingham, 4 pages, 12mo.) (1850?)

Mr. Murphy invites attention and objection to some assertions, as that the earth is prolate not oblate. "If the philosopher's conclusion be right, then the pole is the centre of a valley (!) thirteen miles deep." Hence it would be very warm. It is answer enough to ask—Who knows that it is not?

History of the signs + and —.[*]

Sept. 9, 1864.

I have just traced out a curious point connected with this subject, which, so far as I have read, is wholly unknown to mathematicians. I here give a slight sketch, reserving proofs and authorities for a more appropriate channel of publication.[†] The sign + is read *plus*, which means *more*; and — is read *minus*, which means *less*. The algebraical language 6 + 3, six *more* three, for 9, and 12 — 7, twelve *less* seven, for 5, seems rather forced. It was not so at the outset; + was not a sign of junction, but a memorandum of *more* arrived at than was wanted; and — was the corresponding memorandum of *less*. These memoranda were used in the rule of false position, or *regula falsi* as it was called. In this rule a question is solved, if it lead to a simple equation, by assuming two numbers for the answer, and trying what errors they give: the assumed numbers and the errors they give are made to determine the true answer. But an assumed number may give too much, or too little; and a slight difference of final process depends upon whether both assumed numbers give too much or too little, or else one too much and the other too little. A great many writers made their note of the cases, as they arose, by the letters *p* and *m*, some by *plus* and *minus* at length, or by *piu* and *meno*. Some few Germans used the marks + and —, which may be conjectured to be nothing but the simplest way of making two marks denote *more*, and one mark denote *less*. To conjecture we must probably be reduced: for the first use I have as yet found of the signs has no explanation, and is of 1489. I find it again in 1525 and in 1527: and intermediate instances will probably turn up; the mode of expression continued in use, more or less, until the end of the century. Accordingly, the first meaning of 7 + 5 was "The assumption of 5 gives 7 too much."

The latest historical writers give the invention of + and — to Christopher Rudolf, whose first edition is of 1522 or 1524; some say 1526: I have a suspicion that it was never *printed.* This first edition is now quite unknown. There is a Latin translation (MSS. 365. m. 4; another reference is No. 7,365 of the Latin Manuscripts, 4to.) of 1540 in the Imperial Library at Paris, about which those who describe it do not speak distinctly as to its containing the use of + and — which we make. I suppose they mean to imply the affirmative: but there is this point of doubt. Michael Stifelius—those who wish to delatinize him may choose between

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[*] Though this letter did not form part of the series, yet as it gives some account of a conspicuous paradox, and also illustrates the history of algebra, we have inserted it here.

[†] See *Transactions of the Cambridge Philosophical Society*, vol. xi., part 1.

Stifel, Stiefel, Stiffel, and Stifels, or may invent a fifth choice—published a second edition of Rudolph in 1571, which is quite clear upon the point in question: I have not seen it, but there are good descriptions. This same Stifelius had published his own work on algebra in 1544, in which he makes large use of + and —. Did he adopt the signs for himself, and then introduce them into his edition of Rudolph? Or did he take them into his own work from the first edition of Rudolph? Probably the second; but not certainly. Editors had not our notions of their duty to their authors; and it has happened that later editions have had editors' improvements made to pass for authors' original text. And Stifelius was a queer man. He has introduced into this very work of Rudolph his own interpretation of the number of the Beast. He determined to fix the character on Pope Leo: so he picked the numeral letters from LEODECIMVS, and by taking in X. from LEO X. and striking out M as standing for *mysterium,* he hit the number exactly. This discovery completed his conversion to Luther, and his determination to throw off his monastic vows. Luther dealt with him as straightforwardly as with Melancthon about his astrology: he accepted the conclusions, but told him to clear his mind of all the premises about the Beast. Stifelius did not take the advice, and proceeded to settle the end of the world out of the prophet Daniel: he fixed on October, 1533. The parishioners of some cure which he held, having full faith, began to spend their savings in all kinds of good eating and drinking; we may charitably hope this was not the way of preparing for the event which their pastor pointed out. They succeeded in making themselves as fit for Heaven as Lazarus, so far as beggary went: but when the time came, and the world lasted on, they wanted to kill their deceiver, and would have done so but for the interference of Luther. I cannot trust so eccentric a man as an editor: I think it possible he may have played up old 666 with Rudolph's text. I should hope some one would examine the manuscript above described, and give me some account of the appearance of + and —, and of the way in which they are used.

I lay no stress on the use of + and —, in the first manner above described. Either Rudolph or Stifelius is, so far as known, the real introducer of these signs, as at present used. That they had the sense to pick out the signs from the *regulâ falsi,* and generalize their use, instead of inventing something not so good for the greater glory of their own originality, adds greatly to their merit.

A. DE MORGAN.

Derivation of + and —.

Oct. 4th, 1864.

That + was the corruption of *et* was proposed in the *Magazine of Popular Science,* vol. i., p. 191, by the late T. S. Davies. He was also confident that — was the line drawn above the contracted word *mns.* The late Prof. Rigaud suggested for + a corruption of *p,* but did not take the old form of the letter: he also thought that — was but the last form of *m,* very often seen in the final *m* of hurried handwriting in our own day, in which a dash joined to *hi* does duty for *him.* The + being originally not a sign of addition, as I have shown, but of *more,* the *et* must be abandoned. I think the abbreviations of *p* and *m* very probable: in the old P the upstroke and cross-line are prominent, and the loop is a junction. But with them may compete the notion that two strokes were used for more, and one for less, without any obligation to the alphabet. A great many cases are given in which simple duplication stands for indefinite majority: it was somewhere proposed that, in a personification, eternity should be represented twice as large as time.

A. DE MORGAN.

No. XX. 1851—1854.

1851. The following letter was written by one of a class of persons whom, after much experience of them, I do *not* pronounce insane. But in this case the second sentence gives a suspicion of actual delusion of the senses; the third looks like that eye for the main chance which passes for sanity on the Stock Exchange and elsewhere:—

15*th Sept.,* 1851.

GENTLEMEN,—I pray you take steps to make known that yesterday I completed my invention which will give motion to every country on the Earth;—to move Machinery!—the long sought in vain "Perpetual Motion"!!—I was supported at the time by the Queen and H.R.H. Prince Albert. If, Gentlemen, you can advise me how to proceed to claim the reward, if any is offered by the Government, or how to secure the PATENT for the Machine, or in any way assist me by advice in this great work, I shall most graciously acknowledge your consideration.

These are my convictions that, my SEVERAL discoveries will be realised; and this great one can be at once acted upon: although at this moment it only exists in my mind, from my knowledge of certain fixed principles in nature:—the Machine I have not made, as I only completed the discovery YESTERDAY, Sunday!

I have, &c., —— ——

To the Directors of the London University,
Gower Street.

The Divine Drama of History and Civilisation. By the Rev. James Smith, M.A. London, 1854, 8vo.

I have several books on that great paradox of our day, *spiritualism*, but I shall exclude all but three. The bibliography of this subject is now very large. The question is one both of evidence and speculation;—Are the facts true? Are they caused by spirits? These I shall not enter upon: I shall merely recommend this work as that of a spiritualist who does not enter on the subject, which he takes for granted, but applies his derived views to the history of mankind with learning and thought. Mr. Smith was a man of a very peculiar turn of thinking. He was, when alive, the editor, or *an* editor, of the *Family Herald:* I say when alive, to speak according to knowledge; for, if his own views be true, he may have a hand in it still. The answers to correspondents, in his time, were piquant and original above any I ever saw. I think a very readable book might be made out of them, resembling *Guesses at Truth:* the turn given to an inquiry about morals, religion, or socials, is often of the highest degree of *unexpectedness;* the poor querist would find himself right in a most unpalatable way.

Answers to correspondents, in newspapers, are very often the fag ends of literature. I shall never forget the following. A person was invited to name a rule without exception, if he could: he answered, " A man *must* be present when he is shaved." A lady—what right have ladies to decide questions about shaving?—said this was not properly a rule; and the oracle was consulted. The editor agreed with the lady; he said that "a man *must* be present when he is shaved" is not a *rule,* but a *fact.*

I cut the following from a Sunday paper in 1849:—

X. Y.—The Chaldeans began the mathematics, in which the Egyptians excelled. Then crossing the sea, by means of Thales, the Milesian, they came into Greece, where they were improved very much by Pythagoras, Anaxagoras, and Anopides of Chios. These were followed by Briso, Antipho, [two circle-squarers; where is Euclid?] and Hippocrates, but the excellence of the algebraic art was begun by Geber, an Arabian astronomer, and was carried on by Cardanus, Tartaglia, Clavius, Stevinus, Ghetaldus, Herig*e*nius, Fran, Van Schooten [meaning Francis Van Schooten], Florida de Beau*me*, &c.

Quadratura del cerchio, trisezione dell' angulo, et duplicazione del cubo, problemi geometricamente risolute e dimostrate dal Reverendo Arciprete di San Vito D. Domenico Angherà. Malta, 1854, 8vo.

Equazioni geometriche, estralle dalla lettera del Rev. Arciprete....al Professore Pullicino sulla quadratura del cerchio. Milan, 1855 or 1856, 8vo.

Il Mediterraneo gazetta di Malta, 26 Decembre, 1855, No. 909: also 911, 912, 913, 914, 936, 939.

The Malta Times, Tuesday, 9th June, 1857.
Misura esatta del cerchio, dal Rev. D. Angherà. Malta, 1857, 12mo.
Quadrature of the circle....by the Rev. D. Angherà, Archpriest of St. Vito. Malta, 1858, 12mo.

I have looked for St. Vitus in catalogues of saints, but never found his legend, though he figures as a day-mark in the oldest almanacs. He must be properly accredited, since he has an archpriest. And I pronounce and ordain, by right accruing from the trouble I have taken in this subject, that he, St. Vitus, who leads his votaries a never-ending and unmeaning dance, shall henceforth be held and taken to be the patron saint of the circle-squarer. His day is the 15th of June, which is also that of St. Modestus, with whom the said circle-squarer often has nothing to do. And he must not put himself under the first saint with a slantendicular reference to the other, as is much to be feared was done by the Cardinal who came to govern England with a title containing St. Pudentiana, who shares a day with *St. Dunstan.* The archpriest of St. Vitus will have it that the square inscribed in a semicircle is half of the semicircle, or the circumference $3\frac{1}{5}$ diameters. He is active and able, with nothing wrong about him except his paradoxes. In the second tract named he has given the testimonials of crowned heads and ministers, &c., as follows. Louis Napoleon gives thanks. The minister at Turin refers it to the Academy of Sciences, and hopes so much labour will be judged *degna di pregio.* The Vice-Chancellor of Oxford—a blunt Englishman—begs to say that the University has never proposed the problem, as some affirm. The Prince Regent of Baden has received the work with lively interest. The Academy of Vienna is not in a position to enter into the question. The Academy of Turin offers the most *distinct* thanks. The Academy della Crusca attends only to literature, but gives thanks. The Queen of Spain has received the work with the highest appreciation. The University of Salamanca gives infinite thanks, and feels true satisfaction in having the book. Lord Palmerson gives thanks, by the hand of "William San." The Viceroy of Egypt, not being yet up in Italian, will spend his first moments of leisure in studying the book, when it shall have been translated into French: in the mean time he congratulates the author upon his victory over a problem so long held insoluble. All this is seriously published as a rate in aid of demonstration. If these royal compliments cannot make the circumference of a circle about 2 per cent. larger than geometry will have it—which is all that is wanted—no wonder that thrones are shaky.

(*To be continued.*)

CORRESPONDENCE.

ON THE CONVERSION OF ORDINARY ASSURANCES FOR THE WHOLE TERM OF LIFE INTO ENDOWMENT ONES BY APPLICATION OF BONUS.

To the Editor.

SIR,—As some Offices are in the practice of converting policies, originally effected for the whole of life, into endowment assurances payable on the life assured attaining a certain age or death, by application of the bonus, you may possibly deem the following not unworthy of insertion in the *Journal.*

Let us assume that a person, aged x, effects a policy of £1 for the whole of life, and that at the end of every y years a reversionary bonus, B, is declared, the cash values of which are respectively ${}^1b$, ${}^2b$, ${}^3b$, &c., where ${}^1b$, ${}^2b$, &c., represent the values of the bonus at first, second, &c., investigations. It is required to ascertain at what age the policy would become payable, by applying the values ${}^1b$, ${}^2b$, &c., to convert it into an endowment assurance.

A little consideration will at once show, that, at the end of the first y years, and assuming that the policy can be converted into an endowment assurance payable at the age $(x+y+n)$, or at the end of n years from the present, n must be such that the whole-life premium he at present pays, together with the temporary annuity payable *in advance* for n years, which the surrender value of the whole-life policy at end of y years and the value of the bonus will purchase at the age $(x+y)$, will equal the premium which a person aged $(x+y)$ would require to pay for an endowment assurance payable at end of n years or death.

Now $\dfrac{1}{1+a_x}-d=$ premium for a whole-life policy of £1 payable at death of x.

$1-\dfrac{1+a_{x+y}}{1+a_x}=$ surrender value of such a policy at end of y years.

$1+a_{x+y\,\overline{n-1}|}=$ value of temporary annuity of £1, payable in advance, for n years to $(x+y)$.

$\dfrac{1-\dfrac{1+a_{x+y}}{1+a_x}+{}^1b}{1+a_{x+y\,\overline{n-1}|}}=$ value of temporary annuity, payable in advance, for n years, which the surrender value and value of bonus will purchase.

$\dfrac{1}{1+a_{x+y\,\overline{n-1}|}}-d=$ premium for an endowment assurance of £1, for n years, to $(x+y)$.

We shall therefore have

$$\frac{1}{1+a_x}-d+\frac{1-\dfrac{1+a_{x+y}}{1+a_x}+{}^1b}{1+a_{x+y\,\overline{n-1}|}}=\frac{1}{1+a_{x+y\,\overline{n-1}|}}-d.$$

$$\therefore\ \ 1+a_{x+y_{\overline{n-1}|}}+1+a_x-(1+a_{x+y})+{}^1b(1+a_x)=1+a_x,$$

$$\text{and}\ \ 1+a_{x+y_{\overline{n-1}|}}=1+a_{x+y}-{}^1b(1+a_x).$$

Since, from the tables published, we can readily ascertain the value of $1+a_{x+y}-{}^1b(1+a_x)$, we will get the value of $1+a_{x+y_{\overline{n-1}|}}$; and, by referring to tables of temporary annuities, we will get the age corresponding to this value, and, in consequence, the age at which the policy would become payable. When we have no tables of temporary annuities, we may ascertain the age by the commutation tables thus: by giving to the above formula the D and N values, according to the notation employed by Mr. Chisholm and other writers, we get

$$\frac{\mathrm{N}_{x+y}-\mathrm{N}_{x+y+n}}{\mathrm{D}_{x+y}}=\frac{\mathrm{N}_{x+y}}{\mathrm{D}_{x+y}}-{}^1b\times\frac{\mathrm{N}_x}{\mathrm{D}_x},$$

from which we obtain the value of

$$\mathrm{N}_{x+y+n}=\mathrm{D}_{x+y}\times{}^1b\times\frac{\mathrm{N}_x}{\mathrm{D}_x};$$

and, as the values of the right-hand side of this equation are known, we get N_{x+y+n}; and, by running the eye down the N column, the nearest value to this will represent *nearly* the age at which the policy would become payable by application of the bonus for the first y years.

The value of N_{x+y+n} may be ascertained in a different manner, thus, B being the *reversionary* bonus:—

$$\text{Since}\qquad {}^1b=\mathrm{B}\left\{1-\frac{d\mathrm{N}_{x+y}}{\mathrm{D}_{x+y}}\right\}=\mathrm{B}\left(\frac{\mathrm{D}_{x+y}-d\mathrm{N}_{x+y}}{\mathrm{D}_{x+y}}\right),$$

substituting this value of ${}^1b$ in the equation

$$\mathrm{N}_{x+y+n}=\mathrm{D}_{x+y}\times{}^1b\times\frac{\mathrm{N}_x}{\mathrm{D}_x},$$

we get

$$\mathrm{N}_{x+y+n}=\mathrm{D}_{x+y}\times\mathrm{B}\times\left(\frac{\mathrm{D}_{x+y}-d\mathrm{N}_{x+y}}{\mathrm{D}_{x+y}}\right)\times\frac{\mathrm{N}_x}{\mathrm{D}_x}=\mathrm{B}\times(1+a_x)(\mathrm{D}_{x+y}-d\mathrm{N}_{x+y}).$$

Now, $\mathrm{D}_{x+y}-d\mathrm{N}_{x+y}=\mathrm{M}_{x+y}$ (see *Assurance Magazine*, vol. vi. p. 347),

$$\therefore\ \ \mathrm{N}_{x+y+n}=\mathrm{B}(1+a_x)(\mathrm{M}_{x+y}).$$

This formula will be found more convenient, for practical purposes, than that of Mr. Sprague in the *Magazine* referred to, when the values allowed for surrender of bonus and the premiums are calculated at the same rate of interest.

At the end of another y years, an investigation takes place, at which a cash bonus of ${}^2b$ is declared; it is required to find the age at which the policy would *now* become payable by application thereof.

Let us assume that the policy will become payable at the age of $(x+2y+m)$, or at the end of m years from the present. It will be evident that m must be such that the whole-life premium he at present pays, together with the temporary annuity payable in advance for m years to

$(x+2y)$, which can be purchased by the value of the bonus, the surrender value of the endowment assurance, in the previous case, at end of y years, and the value of the unexpired portion of the temporary annuity (the difference between the endowment assurance and whole-life premiums) in the previous case, will equal the premium for an endowment assurance to $(x+2y)$ for m years. This, when converted into an algebraic form, becomes

$$\frac{1}{1+a_x}-d+\frac{1-\dfrac{1+a_{x+2y\,\overline{n-y-1}|}}{1+a_{x+y\,\overline{n-1}|}}+{}^2b+(1+a_{x+2y\,\overline{n-y-1}|})\left(\dfrac{1}{1+a_{x+y\,\overline{n-1}|}}-d-\dfrac{1}{1+a_x}+d\right)}{1+a_{x+2y\,\overline{m-1}|}}$$

$$=\frac{1}{1+a_{x+2y\,\overline{m-1}|}}-d.$$

$$\therefore\ 1+a_{x+2y\,\overline{m-1}|}+1+a_x-\frac{(1+a_x)(1+a_{x+2y\,\overline{n-y-1}|})}{1+a_{x+y\,\overline{n-1}|}}+{}^2b(1+a_x)$$

$$+\frac{(1+a_x)(1+a_{x+2y\,\overline{n-y-1}|})}{1+a_{x+y\,\overline{n-1}|}}-1+a_{x+2y\,\overline{n-y-1}|}=1+a_x,$$

$$\text{and }\ 1+a_{x+2y\,\overline{m-1}|}=1+a_{x+2y\,\overline{n-y-1}|}-{}^2b(1+a_x).$$

Giving these values, as before, their commutation equivalents, we get

$$\frac{N_{x+2y}-N_{x+2y+m}}{D_{x+2y}}=\frac{N_{x+2y}-N_{x+y+n}}{D_{x+2y}}-{}^2b\frac{N_x}{D_x};$$

$$\therefore\ N_{x+2y+m}=N_{x+y+n}+D_{x+2y}\times{}^2b\times\frac{N_x}{D_x}.$$

It has already been shown that $N_{x+y+n}=B(1+a_x)M_{x+y}$.

Now, $${}^2b=B\left(1-\frac{dN_{x+2y}}{D_{x+2y}}\right),$$

$$\therefore\ D_{x+2y}\times{}^2b\times(1+a_x)=(D_{x+2y}-dN_{x+2y})B\times(1+a_x)=B(1+a_x)M_{x+2y};$$

$$\therefore\ N_{x+2y+m}=B(1+a_x)M_{x+y}+B(1+a_x)M_{x+2y}=B(1+a_x)(M_{x+y}+M_{x+2y}).$$

The age corresponding to the value of N_{x+2y+m} will be the age at which the policy would now become payable.

In a precisely similar manner we may ascertain the age at which another y years' bonus will make policy payable. Assume that it will be at the end of q years from the present, we shall therefore get

$$N_{x+3y+q}=B(1+a_x)(M_{x+y}+M_{x+2y}+M_{x+3y});$$

and so on, until the difference between the ages corresponding to the values of N_{x+ty+s} and M_{x+ty} is less than y, t representing the number of investigations that have taken place.

If the bonus is declared annually, instead of at the end of every y years,

$$N_{x+t+s}=B(1+a_x)\{R_{x+1}-R_{x+t+1}\}.$$

It will be evident from the foregoing, that when we know the age at which the whole-life policy would become payable by application of the bonus, and the age at entry, we can ascertain the rate of reversionary bonus necessary for this purpose.

Since $N_{x+ty+s}=B(1+a_x)\{M_{x+y}+M_{x+2y}+M_{x+3y}+\ldots . M_{x+ty}\}$, dividing both sides of the equation by $(1+a_x)(M_{x+y}+M_{x+2y}+\ldots . M_{x+2y}$, we get

$$B=\frac{N_{x+ty+s}}{(1+a_x)(M_{x+y}+M_{x+2y}+M_{x+3y}+\ldots M_{x+ty})}.$$

As B represents the reversionary bonus for y years, $\frac{B}{y}$ will represent the rate for each year.

The above formulæ will only apply when the reversionary bonus declared at each investigation is the same, when it is different we get

$$N_{x+ty+s}=(1+a_x)\{{}^{1}BM_{x+y}+{}^{2}BM_{x+2y}+{}^{3}BM_{x+3y}+\ldots . {}^{t}BM_{x+ty}\},$$

where ${}^{1}B$, ${}^{2}B$, &c., represent the reversionary bonuses declared at first, second, &c., investigations. This formula will also enable us to find the age at which the policy would become payable when the premiums and cash values of bonus are calculated at *different* rates of interest, as we can readily find what reversionary bonus can be given, at the same rate of interest as that on which the premiums are based, for the cash values of the bonus.

In this latter case, however, possibly the formula, a modification of that of Mr. Sprague,

$$N_{y+ty+s}=(1+a_x)\{{}^{1}b.D_{y+y}+{}^{2}b.D_{x+2y}+\ldots . {}^{t}b.D_{x+ty}\}$$

would be better, ${}^{1}b$, ${}^{2}b$, &c., being the cash values.

As an example of the preceding, let it be required to find the age at which a whole-life policy for £1 on the life of a person aged 20, would be converted into an endowment assurance, by application of the reversionary bonus, declared quinquennially, at the rate of £2 per cent. per annum, Carlisle 4 per cent.

At the end of the first five years we shall have

$$N_{25+n}=B(1+a_{20})(M_{25});$$

$$B=\cdot 1,\quad 1+a_{20}=19{\cdot}3617,\quad \text{and } M_{25}=623{\cdot}86064.$$

$$\therefore\quad N_{25+n}=\cdot 1\times 19{\cdot}3617\times 623{\cdot}86064=1207{\cdot}9.$$

On referring to the N column, we find that 69·9 is the nearest age corresponding to this value, which is the age at which the whole-life policy would become payable by application of the first five years' bonus.

At the second investigation we shall have

$$N_{30+m}=B(1+a_{20})\{M_{25}+M_{30}\}=1207{\cdot}9+1{\cdot}93617\times 545{\cdot}1327$$
$$=2263{\cdot}3705,$$

from which we find that policy would now become payable at (say) 64·8.

At the third investigation we shall have

$$N_{35+s}=B(1+a_{20})(M_{25}+M_{30}+M_{35})=2263{\cdot}3705+1{\cdot}93617\times 468{\cdot}2037$$
$$=3169{\cdot}8931,$$

from which we find that policy would now become payable at (say) 61·6.

At the fourth investigation we shall have

$$N_{40+c}=B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40})$$
$$=3169{\cdot}893+1{\cdot}93617\times 403{\cdot}57=3951{\cdot}2746,$$

from which we find that policy would now become payable at (say) 59·3.

At the fifth investigation we shall have

$$N_{45+s}=B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45})$$
$$=3951{\cdot}27+1{\cdot}93617\times 339{\cdot}12=4607{\cdot}8712,$$

from which we find that policy would now become payable at (say) 57·7.

At the sixth investigation we shall have

$$N_{50+}=B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45}+M_{50})$$
$$=4607{\cdot}871+1{\cdot}93617\times 288{\cdot}67=5166{\cdot}7987,$$

from which we find that policy would now become payable at (say) 56·3.

At the seventh and last investigation, as the difference between the ages corresponding to values N_{55+s} and M_{55} is now less than 5, we get

$$N_{55+s}=B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45}+M_{50}+M_{55})$$
$$=5166{\cdot}7987+1{\cdot}93617\times 248{\cdot}22=5647{\cdot}3987,$$

from which we find that the whole-life policy would, on the assumptions stated, become payable at *nearly* age 55·3.

This result may obviously be verified by ascertaining what endowments, payable at this age, could be purchased by the cash values of the bonus at each investigation, and adding thereto the surrender value of the whole-life policy on the life of 20 at end of 35·3 years.

This rule is derived from the formula

$$N_{x+ty+s}=(1+a_x)\{{}^1bD_{x+y}+{}^2bD_{x+2y}+\ldots\ldots{}^tbD_{x+ty}\}.$$

Dividing both sides of this equation by D_{x+ty+s} and $1+a_x$, changing the signs and adding 1 to each, we get

$$1-\frac{D_x}{N_x}\cdot\frac{N_{x+tv+s}}{D_{x+ty+s}}=1-\left\{\frac{{}^1bD_{x+v}+{}^2bD_{x+2y}+\ldots\ldots{}^tbD_{x+ty}}{D_{x+ty+s}}\right\},$$

$$\therefore\ 1-\frac{D_x}{N_x}\cdot\frac{N_{x+ty+s}}{D_{x+ty+s}}+\frac{{}^1bD_{x+y}+{}^2bD_{x+2v}+\ldots\ldots{}^tbD_{x+ty}}{D_{x+ty+s}}=1 \text{ (the sum assured).}$$

Now, $1-\frac{D_x}{N_x}\cdot\frac{N_{x+ty+s}}{D_{x+ty+s}}$ =surrender value of policy on life of x at the age $(x+ty+s)$, and $\frac{{}^1bD_{x+y}+{}^2bD_{x+2y}+\ldots\ldots{}^tbD_{x+ty}}{D_{x+ty+s}}$ =endowments payable at the age $(x+ty+s)$, which the values of the bonus at each investigation will purchase.

I remain, your obedient servant,

Glasgow, October 24th, 1866. T. M.

NEW TABLES OF ANNUITIES, ETC., BY DR. OTTO BEECK, OF HALLE.

To the Editor.

Dear Sir,—Allow me to draw the attention of the readers of the *Journal* to a set of Tables just published by Dr. Otto Beeck, in Halle. The first set contains the values of annuities during the joint duration of two lives, numbers and logarithms, Experience mortality, interest at 3½ per cent. The tables are completely calculated, not interpolated; they begin with the age of 15 years, and contain all differences of age up to 55 years. They are published with excellent good figures, only on one side printed, to be fixed on cardboard for office use. The price for the 12 tables, with 15 columns each, is 3 th. 15 gr., about 11*s.* sterling. The author intends next to publish six tables with 15 columns each, premiums of assurance on two joint lives, single and annual premiums, price 2⅔ th. (about 8*s.*); then 16 tables with 10 columns each, single and annual premiums for survivorship assurance, price 4⅚ th. (about 15*s.*); and then five tables containing the values to be placed to the reserved fund for the assurance of deferred annuities with single and annual premiums, price 2¼ th. (about 4*s.*).

It depends on the success which these tables meet with, whether the author will continue the publication of similar tables, the computation of which is a laborious task, as most of the readers of this *Journal* well know by their own experience. To these readers I need not add a word as to the value of the tables, which, even to those who have made similar computations will furnish very useful means for checking.*

I am, dear Sir,
Yours most truly,

Hamburg, 17*th November,* 1866. WILHELM LAZARUS.

THE FORMULÆ IN MR. DAVID JONES'S WORK, COMMENTED ON BY MR. MACFADYEN.

To the Editor.

Sir,—In the October Number of the *Journal* there appears a letter from Mr. Macfadyen, of the City of Glasgow Life Assurance Company, pointing out what he considers a serious error in one of the expressions given at p. 170 in Jones's work on *Annuities and Reversionary Payments.* I trust the following explanation may be deemed sufficient to exonerate Jones's work from the charge of error.

The value of the assurance of a sum to be received at the end of the year in which the life or lives shall fail, provided that event takes place after t years, is admitted by Mr. Macfadyen, to be correctly given as

$$A_{\overline{(m,\,m_1,\,m_2,\,\&c.)}\rceil t}^{\;v}=r^{t+1}p_{\overline{(m,\,m_1,\,m_2,\,\&c.)}_t}^{\;v}-(1-r)a_{\overline{(m,\,m_1,\,m_2,\,\&c.)}\rceil t}^{\;v}.$$

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* Before Dr. Beeck proceeds further with his publication, it may be well to draw his attention to the circumstance that new and very extensive data are now being collected and arranged under the direction of the Council of the Institute, as mentioned in their last Report, with a view to the more accurate determination of the probable duration of human life.—Ed. *J. I. A.*

Aware that, by a further transformation, a much more convenient expression applicable to single and joint lives could be got, Mr. Jones gives this in addition; but it would be absurd to suppose that in doing so he was ignorant of the fact that it did not apply to all cases. As a proof of this, he has, in his recapitulation of formulæ, p. 215, only referred to single and joint lives.

Baily, in a Note at page 147 of his work, says, "Since $(\mathrm{ABC})^t$ is, by Prob. I., cor. 3, equal to $\mathrm{A}^o\mathrm{B}^o\mathrm{C}^o \times \frac{\alpha\beta\gamma}{abc}(1+\rho)^{-n}$, it is obvious that the present value, in the case of single or joint lives, might be more conveniently expressed by $s \times \frac{1-\rho\mathrm{A}^o.\,\mathrm{B}^o.\,\mathrm{C}^o}{1+\rho} \times \frac{\alpha\beta\gamma}{abc}(1+\rho)^{-n}$; and it is from this formula that I have deduced the rule in Question 28, Chap. XII. But that rule will not extend to *all* cases."

If any blame at all attaches to Jones, it can only be for his omitting to state that the formula $r^t p_{\overline{(m,\, m_1,\, m_2,\, \&c.)}^{\,v}\,t}\mathrm{A}_{\overline{(m+t,\, m_1+t,\, \&c.)}^{\,v}}$ did not apply to all cases.

I am, Sir,
Your obedient servant,
COLIN McCUAIG,
Assistant Actuary Scottish Union Insurance Company.

Edinburgh, 47, *George Street*,
3rd December, 1866.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

First Ordinary Meeting. Session 1866–7.—*Monday*, 26*th November*, 1866.

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting.

The following gentlemen were elected Associates, viz.:—

Alexander Gordon Brown.
William Braid.
Frederick J. Hallow.
Henry F. W. Cowley.

Mr. T. B. Sprague, M.A., read a paper "On the value of annuities payable half-yearly and quarterly."

Thanks were voted to Mr. Sprague, and the meeting adjourned to Monday, the 17th December.

Second Ordinary Meeting. Session 1866–7.—*Monday*, 17*th December*, 1866.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected members of the Institute, viz.:—

Fellow.

Henry Gamble Hobson.

Associates.

Augustus H. Browne.
Bernard Woods.
Nicholas Hanhart.
Leycester H. Greaves.

A communication from Mr. Meikle, "On the arrangement of the data of Life Assurance Offices," and a paper by Mr. Sprague, "On the limitation of risks" (Part II.), were read.

Thanks were voted to Mr. Meikle and Mr. Sprague, and the meeting adjourned to the 28th January, 1867.

SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS.

In October, 1861, we published solutions by Mr. Sprague of the questions proposed at the second year's examination of the Institute for 1860, and we believe those solutions have been found useful by students preparing for the examination; we therefore propose to publish from time to time the questions for the subsequent years, with brief answers to such as are not taken direct from the text books, and we now accordingly submit the papers for 1861, 1862 and 1863, with such answers, Mr. Sprague having again obligingly supplied the latter. We believe it will be found that these questions have been drawn up with great care, and that there is much in them which deserves the attention, not only of the student, but of all persons interested in the theory of life assurance.—Ed. *J. I. A.*

Questions for 1861.

1. Given—

$$\log_e 2 = {\cdot}693147$$

$$\log_e 3 = 1{\cdot}098612$$

$$\log_e 10 = \frac{1}{{\cdot}434294}$$

Find the values of $\log_{10} 6$, $\log_{10} 8$, $\log_{10} 12$.

To how many decimal places can the results be depended upon?

Ans. $$\log_{10} 6 = \frac{\log_e 2 + \log_e 3}{\log_e 10} = 1{\cdot}791759 \times {\cdot}434294 = {\cdot}7781501$$

$$\log_{10} 8 = \frac{3 \times \log_e 2}{\log_e 10} = 2{\cdot}079441 \times {\cdot}434294 = {\cdot}9030887$$

$$\log_{10} 12 = \frac{2 \log_e 2 + \log_e 3}{\log_e 10} = 2{\cdot}484906 \times {\cdot}434294 = 1{\cdot}0791797$$

These results can only be trusted to six decimal places, and an error of 1 may be expected in the sixth place.

2. There are n balls in a bag, one of them black and the remainder white; and n persons in succession draw each one ball from the bag. Prove that they all have the same chance of drawing the black ball.

3. Give expressions for the probabilities of the various contingencies which may arise out of the occurrences of three independent events, calling the events A, B, C, and the probabilities a, b, c, respectively.

Ans. The chance of ABC all happening $=abc$.
AB happening, C failing $=ab(1-c)$.
A „ BC „ $=a(1-b)(1-c)$.
ABC all failing $=(1-a)(1-b)(1-c)$.

4. An urn contains $n+1$ balls, marked with the numbers 0, 1, 2, 3, . . . n. A ball is successively drawn and replaced in the urn, so that the chance of drawing any given number remains the same in each trial. What is the probability that in h trials the sum of the numbers drawn will be equal to s?

Ans. The probability = the coefficient of x^s in the expansion of

$$\left\{\frac{1-x^{n+1}}{(n+1)(1-x)}\right\}^h.$$

Vide Jones *On Annuities*, *Appx. on Probs.*, p. 12.

5. Find an expression for the number of years in which any sum of money will double itself at compound interest, and prove that it is approximately equal to $\frac{69}{I}$: I being the rate per cent.

Ans. $n=\frac{\log_e 2}{\log_e(1+i)}=\frac{\cdot693147}{i-\frac{i^2}{2}+\dots}=\frac{\cdot69}{i}$ or $\frac{69}{I}$, approximately.

6. Determine the fine which ought to be paid for renewing any number of years lapsed in a lease.

7. Describe the ordinary form of a table of mortality.

8. Given a table of mortality, show how to find (1) at what age it is most probable a person of a given age will die; (2) how many years he has an even chance of living?

9. Give some account of the more recent tables of mortality, and of the materials from which they were constructed (including the published experience of Life Assurance Companies); and point out the uses for which they are most applicable.

10. Supposing a given number of marriages contracted between males of the age of 30, and females of the age of 25, find the proportion per cent. of the original number who will survive as married couples, widowers or widows, at the end of 10 years; assuming the probability of dying within 10 years at the age of 30 to be $\frac{265}{2501}$, and at the age of $25=\frac{237}{2611}$.

Ans. The proportion of married couples $=\frac{2236}{2501}\times\frac{2374}{2611}$; of widowers $=\frac{2236}{2501}\times\frac{237}{2611}$; of widows $=\frac{265}{2501}\times\frac{2374}{2611}$.

11. Describe a practical method of constructing a table of the logarithms of the value of £1, receivable at the end of any number of years, from 1 to 100.

12. Explain briefly the construction and use of the columns D, N, M.

13. Prove that $M_x=vN_{x-1}-N_x$.

14. Find the annual premium for an insurance of £1 on the life of x provided he die within t years.

15. Find an expression for the value of a deferred annuity on two lives

x and y, to commence in t years, and to continue as long as either of the lives is in existence.

16. Find the single premium for the assurance of £1, payable on the death of A aged x, provided he dies before B aged y, or within t years after the death of B.

17. Find the present value of a given sum payable on the decease of A, provided he be the first that fails of the three lives A, B, and C.

18. It is commonly thought that the value of a policy which has been only one year in force, may be found by subtracting the term premium for an insurance for one year, from the net premium for the whole of life. What correction must be applied, in order to give an accurate result?

Ans. The difference so found is to be multiplied by $\frac{1}{vp_x}$, or $\frac{D_x}{D_{x+1}}$.

19. Give a formula for constructing a descending scale of premiums, when the premium diminishes by equal decrements every five years, and becomes zero after 20 years.

Ans. The premium for the first five years

$$=\frac{M_x}{N_{x-1}-\frac{1}{4}(N_{x+4}+N_{x+9}+N_{x+14}+N_{x+19})}.$$

20. Find an expression for the value of an "endowment assurance" (*i.e.*, a policy on a life x payable at age $x+n$, or previous death), after it has been t years in force.

$$\textit{Ans.}\quad 1-\frac{1+{}_{\overline{n-t-1}|}a_{x+t}}{1+{}_{\overline{n-1}|}a_x} \text{ or } 1-\frac{N_{x+t-1}-N_{x+n-1}}{N_{x-1}-N_{x+n-1}}\cdot\frac{D_x}{D_{x+t}}.$$

21. An annuity is granted on the longest of three lives, A, B, and C, to be equally divided between them whilst they are all living; on the decease of either, to be equally divided between the survivors during their joint lives, and then to go to the last survivor during his life.

Give the formula for the value of A's interest therein.

22. Find the value of a reversionary annuity, to commence on the death of x, and continue for the remainder of the life of y; but to be payable for t years, whether y is alive or dead.

$$\textit{Ans.}\quad \left(\frac{1}{i}-a_x\right)(1-v^t)+v^t p_{y,\,t}(a_{y+t}-a_{x.y+t}).$$

23. Find the annual premium for a deferred annuity on a life m, to commence at the end of t years, with the condition that if m die within t years, the premiums paid are to be returned with their accumulations at compound interest.

Ans. The annual premium

$$=\frac{N_{m+t}}{N_{m-1}-N_{m+t-1}-(1+j)M_m-(1+j)^t M_{m+1}\ldots-(1+j)^t M_{m+t-1}+\frac{1+j}{j}\{1+j)^t-1\}M_{m+t}}.$$

24. If the rate at which the premiums are accumulated is the same as that at which they are calculated, show that the expression for the annual premium becomes $\frac{da_{m+t}}{(1+i)^t-1}$; d being equal to $\frac{i}{1+i}$.

25. A debenture for £100, bearing interest at the rate of 6 per cent. per annum, and redeemable in 20 years, is purchased for £107½. Show how to determine approximately the rate of interest realized—given that at 5 per cent. the debenture would be worth £112. 9*s*. 3*d*.

Ans. £5. 7*s*. 7*d*. per cent.

QUESTIONS FOR 1862.

1. Show that in all systems the log. of unity is nothing.

2. Having given the log., to find the corresponding number; or, in other words, expand a^x in a series proceeding by powers of x.

3. If the chance of any one of n persons dying in the course of a year be represented by p, then the chance of exactly r of them dying in the course of a year will be equal to

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} p^r(1-p)^{n-r}.$$

4. On the suppositions in the previous question, prove that the most probable number of deaths in the year is the greatest integer contained in $(n+1)p$.

5. If two joint proprietors have an equal interest in a freehold estate worth £p per annum, but one of them purchase the whole to himself by allowing the other an equivalent annuity of £q for n years; find the relation of p to q.

Ans. $q=\frac{1}{2}\frac{p}{1-v^n}$.

6. If P represent the population of any place at a certain time, and every year the number of deaths is $\frac{1}{p}$th, and the number of births $\frac{1}{q}$th, of the whole population at the beginning of that year, required the population at the end of n years from that time.

7. Given a table of annual premiums; show how to construct a corresponding table of mortality.

Ans. If P_x be the annual premium at the age x, p_x the chance of living a year, then $p_x=(1+i)\left(\frac{1}{P_x+d}-1\right)(P_{x+1}+d)$.

8. Give some account of the Northampton table of mortality, and point out the fundamental error in its construction.

9. The annual mortality at each age being given, show how to construct therefrom a table of the mean duration (or expectation) of life.

10. To which of the ordinary tables of premium published in a prospectus is the Carlisle table of mortality inapplicable as a basis of calculation, and why?

11. Find (by the old method) the value of an annuity on a single life, and show how the annuity at any age may be given in terms of the next higher age.

12. Explain the several columns of a single life commutation table.

13. Describe the actual construction of column D, and point out how the results may be verified.

14. Find the value of an annuity on the survivor of any number of lives—that is, to continue so long as any one of them exist.

15. Prove that if $\frac{1}{4}$ be added to the value of a life annuity of £1 payable yearly, the sum will be very nearly equal to the value of the same annuity payable half yearly.

16. Investigate a formula for the annual premium for a deferred annuity, with a return of all the premiums in the event of previous death.

17. Explain the method of calculating a table of ascending premiums, giving the proper formulæ.

What precaution must always be observed in practice with regard to the magnitude of these premiums, and how would you fix the premium to be charged during the first term of years?

Ans. The premium charged for the first term must not be less than the premium for a term insurance of that duration, and should be taken equal to the term premium for, say, three years older.

18. Prove that the value of £1 to be received on the decease of a person of a given age is greater than that of £1 receivable at the expiration of a number of years certain equal to the expectation of life at that age.

19. Required the present value of a sum of money to be received at the end of the year in which A dies, provided he die while B is living.

20. Give an expression for the value of a contingent reversion expectant on the decease of A, provided B be then living and C dead.

21. Find the value of a contingent life interest for the lives of A and B and the survivor, after the death of C; subject to the condition that A shall survive C.

22. Find an expression for the value of a reversionary annuity of £1 to commence on the death of a person now aged x, and continue payable till the expiration of t years from the present time, and as much longer as a person now aged y shall survive.

Ans. $\dfrac{1-v^t}{i} - {}_{t|}a_x + a_{y\overline{t|}} - a_{xy\overline{t|}}$.

23. Find the amount of an annuity of £1 payable quarterly in advance for n years certain, supposing that the investments made at the beginning of each year are at the rate of £i per £1 per annum payable yearly, and those made in the interval are at the rate of £j per £1 per annum payable quarterly.

Ans. $\frac{1}{4}\left\{1+i+\dfrac{4+j}{j}\left[\left(1+\dfrac{j}{4}\right)^3-1\right]\right\}\cdot\dfrac{(1+i)^n-1}{i}$.

24. If the rate of mortality in one table be throughout greater than that in another table, should you expect the values of policies obtained in the ordinary way from the first table to be greater or less than those obtained from the second? State your reasons.

Ans. No conclusion can be drawn from the simple fact of the rate of mortality being greater.

Prove that, under certain conditions, the values of all policies as given by two different tables of mortality will be equal.

Ans. This will be the case if $\dfrac{1+a_x}{1+\acute{a}_x}$ is constant, or has the same value for all values of x, a_x and $\acute{a}_x$ being the annuities in the two tables of mortality.

25. Find the annual premium for an assurance of £1 payable at the expiration of $m+n$ years if A be then alive, or at the death of A if that

take place after the expiration of m years but before the expiration of $m+n$; with the further condition, that if A die within m years from the present time the premiums paid are to be returned, with the exception of the first.

Ans. $$\frac{M_{x+m}-M_{x+m+n}+D_{x+m+n}}{N_{x-1}-N_{x+m+n-1}-R_{x+1}+R_{x+m}+(m-1)M_{x+m}}.$$

QUESTIONS FOR 1863.

1. Explain what is meant by the common logarithm of a number; and show that if several numbers differ only in the position of the decimal point, their common logarithms have all the same decimal part.

2. Prove the principle of proportional parts in common logarithms; and show that with tables of logarithms extending to seven figures, we may expect to find the number corresponding to a given logarithm correct to seven figures, but not more.

3. From a bag containing 2 guineas, 3 sovereigns, and 4 shillings, a person draws three coins: what is the previous value of his expectation?

Ans. £1. 15*s.* 4*d.*

4. A halfpenny being tossed one hundred times, what is the chance that it will turn up head exactly fifty times?

If head should turn up fifty times running, is it more likely that head or tail will occur in the next trial?

Ans. (1) $\frac{100\times 99\times 98\ldots\times 51}{1\times 2\times 3\ldots\times 50}\cdot\frac{1}{2^{100}}$. (2) If head and tail are equally probable, no conclusion can be drawn from head turning up fifty times running; but if it is not known beforehand that head and tail are equally probable, then it is a fair conclusion that head is more likely to turn up than tail.

5. Compare the chances of throwing a single ace in one trial with two dice, and in two trials with three dice.

Ans. The chances are $\frac{5}{18}$ and $\frac{2975}{5184}$.

6. Find the amount at compound interest in n years, of an annuity certain of £1 payable in advance.

Explain how the amount of such an annuity is to be obtained readily from the ordinary tables of annuities certain.

Ans. (1) $(1+i)\frac{(1+i)^n-1}{i}$. (2) The tables usually give the value of an annuity payable at the end of each year, and the value in question is found by subtracting 1 from the tabular value of an annuity for $n+1$ years.

7. The rate of interest being 5 per cent., payable half-yearly, and the income tax 7*d.* in the pound, find the sum which in two years will amount to £1,000. Given that

$$\log 9{\cdot}08534 = {\cdot}9583408$$
$$\log 1{\cdot}0242708 = {\cdot}01041479.$$

Ans. £908. 10*s.* 8*d.*

8. Explain what is meant by a table of mortality, and describe its most usual and convenient form.

9. If $p_{x|n}$ denote the probability that a person of the age x will live for n years, and p_{x+r} the probability that a person of the age $x+r$ will live a year, prove that

$$p_{x|n}=p_x\cdot p_{x+1}\cdot p_{x+2}\cdot\cdot\cdot\cdot p_{x+n-1}.$$

10. Describe very briefly the methods by which the tables of mortality known as the "Northampton" and the "Experience" were formed; and the materials on which they are based.

11. Point out the chief source of error in the formation of the Northampton Table, and state what has been its effect on the table.

How does this affect the values of policies as found by the Northampton Table?

Ans. Dr. Price assumed that the population was stationary, neglecting the increase of the younger lives by immigration; and the table therefore represents the mortality in the first half of the table as much higher than it really is. The result is that the values of policies found by this table are generally too small.

12. What is the principal objection to the use of the Experience Table of Mortality? Is it of any practical importance?

Ans. That it was based on the observation of policies instead of lives, but this is probably of little consequence.

13. Prove the formulæ—

$$a_x=\frac{vl_{x+1}+v^2l_{x+2}+\ldots\ldots}{l_x}$$

$$\mathrm{A}_x=\frac{vd_x+v^2d_{x+1}+\ldots\ldots}{l_x}$$

14. By means of the formulæ in the last question, show that

$$\mathrm{A}_x=v-(1-v)a_x.$$

15. Describe a practical method of computing a table of annuities certain at any assumed integral rate of interest; explaining how the calculations are to be verified as they proceed.

16. Prove the formula for a contingent assurance—

$$\mathrm{A}_{\underset{1}{\underline{x.y}}}=\tfrac{1}{2}\left(\mathrm{A}_{xy}+\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\right).$$

What supposition is made in obtaining this formula, as to the deaths which take place in the course of any year?

Ans. That they are distributed uniformly over the year, or take place at equal intervals therein.

17. Show that the value of £1 to be received on the failure of the *successive* lives x and y is $\mathrm{A}_x\mathrm{A}_y$. When is the second life supposed to be nominated?

Ans. At the end of the year in which the first life dies.

18. Prove the formula for the value of a policy

$$\mathrm{V}_{x|n}=1-(\pi_x+d)(1+a_{x+n}),$$

and point out the advantage of constructing a table of the values of policies from this formula.

19. Show that if π_x is the annual premium for an assurance of £1 at the age x, then

$$\frac{\pi_x}{\pi_{x-1}}<1+\frac{1}{a_{x-1}}.$$

20. A policy is taken out, at the age x, for £1 payable at the age $x+m$ or previous death. When it has been t years in force, the assured wishes the terms of the policy to be altered, so that the sum assured shall be payable at the age $x+m+n$, or previous death. What premium should be paid for the remainder of the extended term?

Ans. $$\frac{\mathrm{N}_{x+t-1}-\mathrm{N}_{x+m-1}}{\mathrm{N}_{x+t-1}-\mathrm{N}_{x+m+n-1}}\cdot\frac{\mathrm{D}_x}{\mathrm{N}_{x-1}-\mathrm{N}_{x+m-1}}-d.$$

21. Find the annual premium for an insurance on the life x of £1, together with a return of all the premiums paid; the premiums being payable for t years only.

Ans. $$\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}-\mathrm{N}_{x+t-1}-\mathrm{R}_x+\mathrm{R}_{x+t}}.$$

22. A policy being taken out at the age x for the whole of life, what portion of the nth annual premium is required for payment of the current claims, assuming that the rate of interest realized, and the mortality experienced, are the same as those on which the premiums are based?

Ans. $$\left(\frac{1}{p_{x+n-1}}-1\right)(\mathrm{P}_x+d)\,a_{x+n-1},\text{ or }v(1-p_{x+n-1})(1-\mathrm{V}_{x|n}).$$

23. A sum of money (S) is to be applied in the purchase of an annuity on three lives, x, y, z, such that the annual payment while all three are alive shall be A; when one has died, $\frac{4}{5}$A; and when only one survives, $\frac{2}{5}$A. Find the value of A.

Ans. $$\mathrm{A}=\frac{5\mathrm{S}}{2(a_x+a_y+a_z)-a_{xyz}}.$$

24. A debenture for £100, bearing interest at the rate of $4\frac{1}{2}$ per cent., and redeemable in 25 years, is purchased for £96. What rate of interest is obtained on the investment? N.B.—The value of the debenture, calculated at *five* per cent., is £92. 19*s*. 1*d*.

25. There are three Insurance Companies, A, B, C. In A there are 1,000 members, each insured for £100; in B there are 100 members, each insured for 1,000; and in C 10 members, each insured for £10,000. Assuming all the assured to be of the same age, and the tabular rate of mortality among persons of that age to be $1\frac{1}{2}$ per cent. per annum, show how to find the probabilities in the three Companies that claims will arise for £10,000 or upwards, in the course of a year.

Ans. The probabilities are—

In A, $$\frac{\underline{|1000}}{200^{1000}}\left\{\frac{3^{100}\times197^{900}}{\underline{|100}\,\underline{|900}}+\frac{3^{101}\times197^{899}}{\underline{|101}\,\underline{|899}}+\frac{3^{102}\times197^{898}}{\underline{|102}\,\underline{|898}}+\ldots+\frac{3^{1000}}{\underline{|1000}}\right\}$$

In B, $$\frac{\underline{|100}}{200^{100}}\left\{\frac{3^{10}\times197^{90}}{\underline{|10}\,\underline{|90}}+\frac{3^{11}\times197^{89}}{\underline{|11}\,\underline{|89}}+\frac{3^{12}\times197^{88}}{\underline{|12}\,\underline{|88}}+\ldots+\frac{3^{100}}{\underline{|100}}\right\}$$

In C, $$1-\left(\frac{197}{200}\right)^{10}.$$

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Arrangement of the Data furnished by certain Life Assurance Companies in Scotland, and on the formation of Tables, &c., therefrom; being Contributions towards the ascertainment of the Rate of Mortality of Assured Lives. By JAMES MEIKLE, *Esq., Actuary of the Scottish Provident Institution.*

[Read before the Institute, 17th December, 1866.]

THE mortality statistics of the Scottish Assurance Offices, whose experience it is proposed to develop before combining it with that of the English contributing Offices, have, after passing through various committees, and receiving the benefit of their consideration, eventually passed into my hands for final adjustment and development. Having accordingly considered the subject thus confided to my professional care, I venture to lay before the Institute a brief statement of the views which have occurred to me in regard to the processes which it is proposed to adopt for the arrangement of these data and transforming them into that shape which it is necessary they should assume before becoming the subject of comparison or of calculation.

The data of each Scottish contributing Company have, it is well known, been furnished upon cards, and consist of the name of the life and the amount assured, the date of birth, the date and age of entry, and the date and age at exit—that is, at the termination of the risk, which may either be that of cancelment, death, or that of the date of investigation, viz., 31st December, 1863.* The cards of deaths were marked by a D, on a special part of the card; and the cancelled by either an S (the initial letter of surrendered) or an L (lapsed); and those which were in existence at 31st December, 1863, were marked by a light stroke, thus ——. White cards were used for males, and pink for females.

Under the impression that experience in this matter was of the greatest importance, I made several experiments with the cards of my own Office, and with those of the Offices which were earliest placed in my hands. The result of these experiments was such as I have every reason to believe, that the process followed with the smaller number of cards of one Office might with at least equal advantage be applied to the entire cards of the ten Scottish contributing Offices. On examining the processes which had been followed in similar investigations based upon the card system, it occurred to me that considerable difficulty would attend the counting and balancing the cards for so large a number as 120,000 lives, when in the first place arranged according to *ages at entry*, and in the second rearranged according to ages *at exit*. I have, however, fallen upon a plan for classifying the cards in small numbers which suits admirably; and not only does it suit in regard to the mere matter of *counting* the cards, but it affords the means of *checking* the accuracy of the calculations, and it also affords the means of showing the *effects of selection*, and the depreciation of life after the benefits of selection have worn off. All of these I shall now try to explain.

It was at first thought that *one* card, representing the continuous experience for each life, would be sufficient, and printed instructions were issued to that effect. I perceived, however, that it was quite possible for *one* life to embody the facts of many experiences, each of which would be represented by each of the policies. These earliest instructions were accordingly altered, and I asked from the Offices a card for each policy except in those cases where they were opened at the same date and were in other matters identical. It was customary, previous to the change in the stamp laws, for one life to open two policies for half the sum assured, for the purpose of saving the difference between the stamp for the full amount and the two stamps for the two halves. In any such case *one* card would sufficiently represent the experience. And in other cases, nearly similar perhaps, one card would be sufficient. At all events the Offices have not been requested to curtail the number of cards; I have rather asked a card for each policy. Each Office was further asked to note the numbers of all the policies on the same life on each card, so as to act as a guide and check in collecting the total number of cards issued on each life. This has been very carefully done by one or two Offices, but others have made no attempt in the matter, and have thereby greatly increased the consequent labour.

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* If there had been any intention of continuing the experience of the same number of entrants, and the opportunity seems an excellent one, it would only have been necessary to have noted the issue of those policies which had terminated by cancelment or by death. Each of the contributing Offices would thereafter only be required to give a note on slips of paper of the policies which had, out of the entrants previous to 1864, become cancelled or dead.

When put in possession of the cards of any Office my first step was to classify them according to the first letter of the *name*. When the cards were thus sorted they were carefully *banded* and laid aside. Each letter was then taken up by itself, and classed according to the *second* letter, and also to the *third* letter; any error in the first or second arrangements being checked in the second or third. Each of these small sections was then taken up and the cards classified according to surnames and to christian names, and thereby the cards belonging to each life were brought together. The *date* of birth being given in the Scotch cards, I judged it to be quite conclusive when the names and the dates of birth were the same, that they referred to the same life. There have been curious instances of approaching similarity, but of different identity, which, if I had had time, it would have been interesting to have noted. It is also most important that a lady's maiden name should be given along with her husband's. There are other modes of recognising the identity than the date of birth. The date of death, or the cause of death; or one policy may be opened on the same date as the lapsing of a former policy, or the same kind of climate extra, or other points which occur to the mind of the classifier. When it was suspected that two or more cards referred to the same life, but which differed in some essential point, they were taken to the Office for adjustment or correction. In place of writing out a new card for each life, I instituted an envelope capable of containing the cards referring to the same life, and printed in the same way as the cards. On the envelope the facts of the *continuous* experience of the life were noted. When the risks on the life were not continuous, the cards were dismissed with a jotting referring to each other, thus—

$$\text{Previous risk, } \frac{35}{40};$$

$$\text{and Risk repeated, } \frac{45}{50};$$

in case it might be necessary to collect these cards again. The flap of the envelope was carefully laid inside, so as to be out of the way in sorting, and at same time so as to afford access to the separate cards.

When the whole cards were gone over they were redistributed according to the *ages at entry,* and it is the distinguishing feature of my process to retain them in this form for all further use. In the first place it is important to make sure that the cards of each of these lots refer to the same age at entry. A check on this is obtained in the second distribution—by making the distributor look at *two* things—that is, in distributing them according to — (existing), L (lapsed), or D (dead), to see that they are all

35 —,
35 L,
or 35 D.

I even recommend that the figures be spoken, which I imagine adds to the security of the check, although it is not an infallible safeguard. But as the process of classification of the same lot of cards has to be performed so frequently, there is no risk of any continued error. The cards of the entrants at each age are now in three divisions—the living, the dead, and the discontinued, the sum of which is the number of the entrants at that age. Each parcel is banded, and the three are again banded in one with a stronger band.

The next part of the process is that which may be called "tabulating the facts." The parcel of cards belonging to each age at entry is taken up, and each of its parts classified separately and tabulated in the following form. (See next page.)

The classification is checked thus:—

The first arrangement is in accordance with the *age at exit.* The cards of each parcel are gone over, counted, and tabulated. The same lot are gone over again while lying on the desk, but while counting their number with the lips the eye this time rests on the fact whether all those who exit at $x+0$ have 0 for the years of duration, and those who exit at $x+1$ have all 1 for years of duration, &c. &c. I have found this check a very valuable one, measuring its value by the number of errors discovered. The same process is gone over for the existing, discontinued, and the dead of each age at entry, and the number of entrants—$m_x - (\Sigma c_{x+n} + \Sigma d_{x+n} + \Sigma l_{x+n})$ is the result of a *summation of figures in ink,* which is much more to be relied on than the more ethereal one of counting cards

Difference between Year of Entry and Year of Exit.	Age attained in Year of Exit.	Number of Lives entered on Risk.	Of whom			Sum of Numbers in Cols. 4, 5, 6.	Sum of Col. 7.	Half of the Number entering, less half the Cancelled.	Completed Age on entering Year of Risk.	Number exposed to Mortality.	Number of Deaths.
			Cancelled	Died	Living at 31st Dec., 18—						
			at Age in Col. 2.								
n.	$x+n$.	m_x.	c_{x+n}.	d_{x+n}.	l_{x+n}.	$(c_{x+n}+d_{x+n}+l_{x+n})$	$\Sigma(c_{x+n}+d_{x+n}+l_{x+n})$.	$\frac{m_{x+n}-c_{x+n}}{2}$.	$x+n-1$.	Col. 8 + Col. 9.	Col. 5.
1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
0	30	1,083	5	3	78	86	..	+541·5 − 2·5	29	539·	3
1	31	..	88	2	62	152	997	−44·	30	953·	2
2	32	..	54	9	65	128	845	−27·	31	818·	9
3	33	..	45	5	67	117	717	−22·5	32	694·5	5
4	34	..	26	2	55	83	600	−13·	33	587·	2
5	35	..	18	1	59	78	517	− 9·	34	508·	1
6	36	..	12	5	51	68	439	− 6·	35	433·	5
7	37	..	29	2	35	66	371	−14·5	36	356·5	2
8	38	..	5	3	42	50	305	− 2·5	37	302·5	3
9	39	..	1	2	37	40	255	− ·5	38	254·5	2
10	40	..	1	3	37	41	215	− ·5	39	214·5	3
11	41	..	..	3	34	37	174	..	40	174·	3
12	42	..	2	2	42	46	137	− 1·	41	136·	2
13	43	..	1	..	32	33	91	− ·5	42	90·5	..
14	44	..	2	1	25	28	58	− 1·	43	57·	1
15	45	..	..	1	15	16	30	..	44	30·	1
16	46	..	..	1	5	6	14	..	45	14·	1
17	47	..	..	..	2	2	8	..	46	8·	
18	48	..	..	..	4	4	6	..	47	6·	
19	49	..	..	..	1	1	2	..	48	2·	
20	50	..	..	..	..	..	1	..	49	1·	
21	51	..	..	..	..	..	1	..	50	1·	
22	52	..	..	..	1	1	1	..	51	1·	
		1,083 =	289	+45	+749	1,083	5,784	+541·5	=	6,181	45
						$\frac{1083-5}{2}=+539$		−144·5			
						$\frac{289-5}{2}=-142$		+397			
							397		=	6,181	

Note.—Since the above was written I have thought it right to add two columns rearranging the deaths, so as to be able to compare the possible years of life enjoyed with the years actually enjoyed.

in the air. The materials have thus been obtained for ascertaining the number exposed to risk of mortality out of all those who entered at each age. This is worked up from the formula—

$$\left.\begin{array}{r}\text{Number exposed to mortality}\\ \text{at completed age }(x+n-1)\end{array}\right\} = m_x - \Sigma_{x+n-1}(c_{x+n}+d_{x+n}+l_{x+n}) + \frac{m_{x+n}-c_{x+n}}{2}$$

$$\text{and } \frac{\text{deaths }(x+n-1)}{\text{No. exposed }(x+n-1)} = \left\{\begin{array}{l}\text{Rate of mortality from}\\ \text{age } x+n-1 \text{ to age } x+n.\end{array}\right.$$

The summation of the number exposed for each age at entry is checked from the totals of the other columns, as shown in the foregoing example.

The tables in their present shape appear to me to be most valuable. They exhibit the light mortality following selection, and also the heavier mortality of lives long exposed—two points which I have had very great anxiety to see developed. I imagine, however, that for both objects these partial tables are incomplete, as afterwards referred to; and that at their present stage they are only parts of an entire average table of mortality for all ages, such as it has hitherto been the practice to develop. I do not doubt that it is possible to deduce *one* table which shall fairly represent assured-life experience based upon an average rate of mortality for each age of life, but I intend to be sceptical upon the point, and to observe how widely the actual progressive rate of mortality deviates on either side of the average, and then to make up my mind whether it would be prudent to frame tables of annuities, &c., applicable to each age of entry. Hitherto I have not seen any grounds for believing that the rate of mortality is to be considered as *constant* at each age. It is exceedingly convenient to make that assumption, but I think it is doubtful whether it is safe to do so, especially when taken in connection with some of the present modes of withdrawing profit. However, this doubt, it will be understood, is, in the meantime, a crochet of mine, as well as, I believe, of some others.

I have now to explain how I propose to frame a mortality table from these minor or partial tables. The numbers *entering* at each age in the partial tables are the same as those *entering* in the complete or average table; but for the number discontinued, died, or living in the average table, the items of these classes are, on three large sheets of paper ruled conveniently for the purpose, recast according to the ages attained in year of cancelment, year of death, or at 31st December, 1863, if surviving; the form of which is obvious, but I annex one as a specimen:—

Discontinued—Recast according to Ages at Exit.

Current Age at Entry.	No. Cancelled.	Current Age at Exit																
		20.	21.	22.	23.	24.	25.	26.	27.	28.	29.	30.	31.	32.	33.	34.	35.	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
20	127	1	38	59	9	6	8	2	4	:	:	:	:	:	:	:	:	
21	169	..	7	112	17	8	11	5	3	8	:	1	:	:	:	:	:	
22	170	..	..	4	56	34	28	13	11	10	8	3	:	1	:	1	:	
23	168	..	..	..	7	55	40	26	14	7	4	6	2	3	3	..	1	..
24	210	..	..	..	..	4	86	33	28	23	13	8	4	5	2	..	3	..
25	222	..	..	..	..	..	6	67	37	32	21	20	13	13	4	6	2	..
26	226	..	..	..	..	..	..	4	83	39	34	10	18	7	12	6	3	..
27	266	..	..	..	.	..	..	..	7	86	48	27	26	19	16	16	7	..
28	263	..	..	..	..	..	..	..	..	8	90	49	30	18	12	20	16	..
29	282	..	..	..	..	..	..	..	..	..	1	81	60	38	18	16	18	..
30	289	..	..	..	..	..	..	..	..	..	..	5	88	54	45	26	18	..
31	275	..	..	.	..	..	..	..	..	..	..	..	2	86	65	34	18	..
32	283	..	..	..	..	..	..	..	..	..	..	..	..	4	73	48	43	..
33	276	..	..	..	..	..	..	..	..	..	..	..	..	..	5	89	52	..
34	290	..	..	..	.	..	..	..	..	..	..	..	..	..	..	7	85	..
35	276	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	5	..
36	271																	
37	277																	
38	264																	
No. discontinued at each Age					102	119	184	158	188	212	219	210	243	248	255	269	271	

These are summed up, and the totals made to agree with the totals of the partial tables. The same process having been gone over with the deaths and the living, the materials are obtained for working up the average or entire table of mortality after the same formula—

$$\text{Number exposed at age } x-1=\Sigma_{x-1}m_x-\Sigma_{x-1}(c_x+d_x+l_x)+\frac{m_x-c_x}{2},$$

which may be further checked by recasting the number exposed to risk for each of the minor tables. This completes the labour, and it may with confidence be maintained that the average table, as well as set of minor tables, are correct.

Some interesting checks may be obtained by preparing tables based upon the totals of the average or complete table and the totals of the partial or minor tables, which will prove eminently satisfactory and complete.

Thus, by forming the products of xm_x, xc_x, xd_x, and xl_x, then $\frac{\Sigma x.m_x}{\Sigma m_x}$ = average age at entry of all,

$\frac{\Sigma x.c_x}{\Sigma c_x}$ = average age at entry of the policies;

and other results which are quite apparent to the ordinary computer.

I.—*Extract from complete Table, deduced directly from the Entrants, Discontinued, Died, and Existing at each Age.*

Current Age at Entry.	m_x.	c_x.	d_x.	l_x.	$c_x+d_x+l_x$.	$\Sigma_{x-1}(c_x+d_x+l_x)$.	$\Sigma_{x-1}m_x$.	Difference.	$\frac{m_x-c_x}{2}$.	Number exposed.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
26	710	158	14	181	353	1,970	3,529	1,559	276·	1835·
27	878	188	16	266	470	2,323	4,239	1,916	345·	2261·
28	910	212	28	286	526	2,793	5,117	2,324	349·	2673·
29	1,000	219	24	332	575	3,319	6,027	2,708	390·5	3098·5
30	1,083	210	22	339	571	3,894	7,027	3,133	436·5	3569·5
31	1,059	243	31	379	653	4,465	8,110	3,645	408·	4053·
32	1,073	248	41	428	717	5,118	9,169	4,051	412·5	4463·5
33	1,123	255	27	489	771	5,835	10,242	4,407	434·	4841·

II.—**Extract from complete Table, deduced from the minor Tables recast.*

Years of Assurance.	PASSING THROUGH AGE									
	27.		28.		29.		30.		31.	
	No. Exposed.	Deaths.	No. Exposed.	Deaths.	No. Exposed.	Deaths.	No. Exposed.	Deaths.	No. Exposed.	Deaths.
0	435·5	1	451·	..	499·5	5	539·	3	528·5	2
1	618·5	5	765·	9	775·	4	885·5	5	953·	2
2	511·5	6	499·5	6	635·	7	631·5	4	767·	8
3	256·	..	411·	4	412·	4	540·5	4	528·	7
4	169·	1	205·5	1	342·5	2	344·	3	455·	9
5	128·5	1	133·5	4	171·5	1	281·	..	301·	
6	49·5	1	99·	1	104·	..	139·	1	237·5	1
7	29·	1	33·5	1	75·	..	87·	1	119·	1
8	19·	..	25·	2	30·	..	57·5	1	69·	
9	11·	..	18·	..	20·	..	26·5	..	49·	1
10	8·5	..	7·	..	12·	1	16·	..	21·	
11	16·	..	8·	..	5·	..	9·	..	13·	
12	6·	..	14·	..	7·	..	4·	..	7·	
13	3·	..	1·	..	8·	..	4·	..	1·	
14	..	..	2·	..	1·	..	3·	..	2·	
15	..	..	..	..	1·	..	1·	..	1·	
16	..	..	..	..	..	..	1·	..	1·	
Totals	2261·	16	2673·	28	3098·5	24	3569·5	22	4053·	31

It will thus be observed that little risk has been incurred in counting the cards. Being divided into comparatively small sections, the risk of error at first was not great: and after the counting had been repeated with reference to the years of duration, the risk almost disappears; and it will further be observed that I do not require to count the cards specially for the *entrants*. There is also, in many of the minor checks, and especially in the check obtained by the recasting of the tabulated facts of the minor tables, a charm imparted to the calculation which excites the computer and carries him on with his work; and if the value of the minor tables be taken into consideration, I believe there is no increase of the labour; at all events it has appeared to me so insignificant as not to give me one moment's consideration.

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* The complete table, of which No. 2 is an extract, is in possession of the Institute.

I shall now point out the use I intend to make of the separate cards in the envelopes. As before referred to, I intend preparing a table of mortality for each age at entry, so far as the facts will avail, and thereafter, by means of some general assumption, to carry them on to the end of life. The minor tables before alluded to represent the experience of *lives* arising out of their *first policies;* and in that light they may be accepted by some actuaries as suitable for some kinds of assurance purposes. But I imagine that if any person, who assured at age x, assures again at $x+5$, that person is again a selected life at $x+5$, and ought also to appear among entrants of $x+5$, and the issue of each selection or entry appear in its own class; and so on at each of the ages at entry, so long as the same life does not appear twice in the same class. The separate cards in the envelopes will be taken out and placed each among entrants of the same age, and the facts retabulated, for the special purpose of showing the rate of mortality of entrants of each age. These tables are in themselves final. They cannot be recast for any average table except as a matter of fancy, or as showing the mortality among policies. They may, however, be recast according to years of assurance, either in decennial, quinquennial, or triennial periods, or the whole tables thrown together in one, and would, I imagine, present some very important features of mortality risks.

I annex two specimens, based upon the A_1, A_2, A_4, and A_5 of Experience Tables. The *first* shows the number exposed to mortality in each year of assurance, passing through the two quinquenniums 35—40 and 40—45, together with the computed and the actual deaths in the same period of life. The *second* specimen throws the whole number (31,380) into years of assurance. The computed mortality is based upon the ratio of the number dying to the number exposed in each year of life in the usual way. Both tables illustrate very forcibly that the excess of the computed deaths in the early years of assurance (profitable) is balanced by the deficiency of the computed deaths in after years (unprofitable), and appear to me to possess great interest. They seem to indicate that life assurance valuations based upon an *average* table of mortality would require to be "corrected for selection."

Calendar Years of Assurance.	35–40.					40–45.				
	Number exposed.	Actual No. of Deaths.	Computed No. of Deaths.	Computed No. in excess.	Actual No. in excess.	Number exposed.	Actual No. of Deaths.	Computed No. of Deaths.	Computed No. in excess.	Actual No. in excess.
0	2626·5	17	26·4	9·4	..	2159·5	7	25·1	18·1	
1	4717·5	39	47·8	8·8	..	4036·5	37	46·9	9·9	
2	4142·	39	42·0	3·0	..	3462·	36	40·2	4·2	
3	3573·	27	36·1	9·1	..	3023·	33	35·0	2·0	
4	3033·5	33	30·6	..	2·4	2699·	37	31·5	..	5·5
5	2622·	33	26·4	..	6·6	2438·5	29	28·3	..	·7
6	2194·	32	22·0	..	10·	2136·	34	24·8	..	9·2
7	1796·	23	18·1	..	4·9	1889·5	30	22·0	..	8·
8	1417·	16	14·3	..	1·7	1540·5	20	17·8	..	2·2
9	1124·5	12	11·3	..	·7	1269·5	19	14·7	..	4·3
10	840·	10	8·4	..	1·6	1080·5	15	12·6	..	2·4
11	605·	6	6·0	..	..	880·5	11	10·2	..	·8
12	389·5	4	3·8	..	·2	774·5	13	7·8	..	5·2
13	178·	2	1·7	..	·3	416·5	1	4·8	3·8	
14	71·	2	·7	..	1·3	210·5	2	2·4	·4	
15	29·5	1	·3	..	·7	95·	1	1·1	·1	
16	9·	0	·1	·1	..	38·	1	·4	..	·6
17	2·	0	·0	·0	..	15·	0	·2	·2	
18	..	..	..	..	..	9·	0	·1	·1	
19	..	..	..	..	..	4·	0	·1	·1	
20	..	..	..	..	..	3·	0	·0		
21	..	..	..	..	..	2·	0	·0		
Totals	..	296	296·	30·4	30·4	..	326	326·	38·9	38·9

Calendar Years of Assurance.	Number Exposed.	Actual Deaths.	Computed Deaths.	Difference.		Total future actual Deaths.	Total future Deaths provided for. Computed.	Difference. The future actual Deaths being in excess of the computed by
				Computed Deaths in excess of actual.	Computed Deaths short of the actual.			
0	15605·5	108	189	81	..	2,482	2,482	0
1	27300·	302	342	40	..	2,374	2,293	81
2	22819·	307	296	..	11	2,072	1,951	121
3	19147·5	256	258	2	..	1,765	1,655	110
4	16155·	225	229	4	..	1,509	1,397	112
5	13860·5	232	205	..	27	1,284	1,168	116
6	11719·5	198	180	..	18	1,052	963	89
7	9778·	182	157	..	25	854	782	72
8	7944·	144	134	..	10	672	625	47
9	6648·	124	115	..	9	528	491	37
10	5485·	100	102	2	..	404	376	28
11	4445·5	86	86	..	..	304	275	29
12	3538·5	90	72	..	18	218	189	29
13	2291·	42	50	8	..	128	117	11
14	1244·5	34	29	..	5	86	67	19
15	653·	22	16	..	6	52	38	14
16	292·5	15	7	..	8	30	22	8
17	173·	5	5	..	0	15	15	0
18	107·	4	3	..	1	10	10	0
19	79·	2	2	..	..	6	7	− 1
20	57·	3	2	..	1	4	5	− 1
21	43·	..	1	1	..	1	3	− 2
22	33·	1	1	..	..	1	2	− 1
23	21·	..	1	1	..	..	1	− 1
Totals	169440·	2,482	2,482	139	139			

With reference to the *number of classifications* of the cards, I should like to see the following, but it is feared that I have proposed more labour than I shall conveniently be able to overtake:—

1. The entire cards.
2. The healthy
3. The diseased } entire cards.
4. The males
5. The females } entire cards.
6. With profits
7. Without profits } entire cards.
8. Those who have incurred risk of foreign climates.
9. Those who have not.
10. Above £1,000.
11. Under £1,000.
12. First year of assurance.
13. Experience of each age at entry.
14. Classification of the partial risks, in envelopes.

I should mention, however, that I have seen it necessary to make a separate class of earls, honourables, or peers' sons, or others who are likely to change their names in the course of the statistical period; and in the case of females it would be right to reclassify the cards according to the date of birth, although even that is not altogether a fixed point in a lady's history, but it may operate as a check against the double name to the same risk. Further, I would propose that the *deaths* be *reclassified* according to date of death, so as to make sure that no death has been counted twice—a matter of more importance than an open risk counted twice.

There are other matters to which I might allude, but in the meantime I confine myself to one point, viz., the desirableness of making no comparisons with tables calculated by methods which might, in the mere matter of calculation, give different results. That the *comparison* of *mortality* be complete, every other point ought to be the same—method of reckoning the ages, method of tabulating the facts, method of interpolation, &c. Overlooking these differences, a comparison may, of course, be made with any table, but it would be better if the tables were deduced by the same process.

On the Mortality in the United States of America, as deduced from the last Census, in 1860. *By* Samuel Brown, V.P.S.S., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 25th February, 1867.]

THERE is no country in which, from a combination of circumstances, a census, if accurately taken, is more likely to afford novel and interesting materials than the United States of America. The marvellous progress of the nation, which, in the last 70 years, has increased in numbers from less than 4 millions to nearly 31½ millions in 1860, is alone sufficient to suggest changes in population, in wealth, commerce, cultivation of land, means of transport and circulation, growth of towns, &c., which, if the facts were carefully examined, would throw light upon many obscure questions in social and political economy, and even by the disturbance of their ordinary laws modify and correct the theories which have been mooted.

But the question of population is more than sufficient for our present inquiry, whilst at the same time it stands at the head of all others for its importance and interest. I have, therefore, taken some pains to examine and analyze the volume showing the final exhibit of the Eighth Census of the United States, which was published last year at Washington, under the direction of the Secretary of the Interior.

As far as regards the population and mortality, the preparation of the returns was committed to the care of Dr. Edward Jarvis, a distinguished writer on statistics, of great ability, and a Corresponding Member of the Statistical Society of London. Considering the difficulties of his task, and the imperfections of his data, which I shall have to point out, he has made the most of the materials at his command, and suggested at least reflections on many novel topics worthy of further research.

Population returns.—The schedules of population were obtained by 4,414 marshals, in the summer of 1860, at the houses of the families in the whole country. They were digested and put in order in the Census Office, and the tables then referred to Dr. Jarvis for his Report thereon. By him the facts were analyzed and new combinations formed. We conclude from this general description that the schedules were not filled up with the statements of the existing population on any given day or night; but that the inquiries were conducted from house to house, as was most convenient to the officers, in the course of the summer. This is a primary and important defect in the collection of the returns, for in a population so shifting and changeable, very great differences might be found, even in the course of a few weeks, in the various States. It would be impossible to prevent individuals, and even whole families in some cases, being enumerated twice or more times by different officials; and others, who may, move into a locality which has been already visited, being wholly omitted. It is possible that the omissions may balance the duplicate returns; but a census should be a collection of facts, not conjectures. No doubt the difficulty would be very great of obtaining a simultaneous numbering of the people of a country of such vast extent and in many districts so thinly peopled, requiring often many miles to be traversed for a few isolated facts. But this is a mere question of expense, and of increasing the number of marshals; and the Government seems by no means niggardly in carrying out this great work, since the appropriation for the purpose, in 1860, was no less than two millions of dollars.

Division into districts.—In order the better to compare the results, the whole country has been divided into nine large districts, arranged according to their geographical position and climatic character. They are as follows:—

District I. Maine, New Hampshire, Vermont, Massachussets, Rhode Island, Connecticut, and New York.
„ II. Michigan, Wisconsin, Minnesota, and Nebraska.
„ III. New Jersey and Pennsylvania.
„ IV. Ohio, Indiana, Illinois, Iowa, and Kansas.
„ V. Delaware, Maryland, District of Colombia, Virginia, and North Carolina.
„ VI. Kentucky, Tennessee, and Missouri.
„ VII. South Carolina, Georgia, Florida, and Alabama.
„ VIII. Mississippi, Louisiana, Arkansas, and Texas.
„ IX. California, Oregon, Washington, New Mexico, Utah, Dacota, and Nevada.

The United States extend from 25° to 49° north almost as far south as the torrid zone. In this vast extent of territory, with low-lying swampy lands on the coast and mountains thousands of feet high in the interior, in regions cultivated or in a state of wild nature, drained or moist, in plains exposed to winds or in sheltered valleys, under the burning sun of the south to the mild and sometimes severe temperature of the north, every variety of climate, and, as regards the mortality, every disease that afflicts human nature may be expected to be found. In this arrangement of districts the object has been to bring together such States as most nearly accord in their climates. Even then the range is great. In I., Maine and New York, the range in the mean temperature is 75°; from 6° in January to 81° in July. In V., Maryland, 60°; from 24° in January to 84° in July. And in VII., South Carolina and Georgia, 50°; from 37° in December to 87° in July.

Population in districts.—The following table shows the white and coloured population in the nine districts according to the Census 1860. (See next page.)

By the last three columns will be seen the proportion which the coloured population (including in that term, as I understand it, blacks and mulattoes) bears to the white population in each district, the numbers given being the proportion to 10,000 males, 10,000 females, and 10,000 of both sexes in each district. The smallest proportion is in II., Michigan, &c.; and the larger proportion in VII., South Carolina, Georgia, Florida, and Alabama, where the coloured are not far short of equalling the white population in number, and in VIII., Mississippi, Louisiana, Arkansas and Texas, where they amount in number to nearly three-fourths of the whites.

Increase of population in districts.—The districts, of course, differ very much in other respects in the nature of their respective populations, especially as to the ages into which the inhabitants are distributed. Some have been inhabited for more than 200 years, others have scarcely been settled for a few years, quite within the present generation. It is only in the former that we can look for the due proportion of people at all ages which are found in the old States of Europe, and then only if they do not continue to be disturbed by the admission of immigrants from Europe or from the other American States. If the immigration is numerous, there is naturally an excess of population in the manhood period of life. There the proportion of young children is small, and of aged persons, compared with the totals, smaller still. After a time, if the immigration ceases or diminishes, the early marriages of men in the prime of life lead to a great and unusual proportion of infants and young children to the whole population; whilst the parents having entered the population not at birth but in advanced life, tend to increase the proportion of the older ages sooner than they would in the old populations of Europe.

Some of the States have increased with extraordinary rapidity in the last ten years, both by immigration and by births from recent marriages, thus showing a great preponderance of numbers

A, *showing the White and Coloured Population of the United States in Districts.*

District.	White.			Coloured.			Proportion of Coloured to every 10,000 White in each District.		
	Males.	Females.	Persons.	Males.	Females.	Persons.	Males.	Females.	Persons.
I. Maine, New Hampshire, Vermont, Massachussets, Rhode Island, Connecticut, New York	3,441,151	3,500,919	6,942,070	34,897	38,819	73,716	101	110	108
II. Michigan, Wisconsin, Minnesota, Nebraska	902,708	805,218	1,707,926	4,387	3,930	8,317	48	48	48
III. New Jersey, Pennsylvania	1,750,676	1,745,282	3,495,958	38,785	43,482	82,267	221	248	235
IV. Ohio, Illinois, Indiana, Iowa, Kansas	3,176,693	2,949,285	6,125,978	28,894	28,531	57,425	90	96	93
V. Delaware, Maryland, District of Colombia, Virginia, North Carolina	1,174,875	1,169,636	2,344,511	559,275	558,228	1,117,503	4,760	4,772	4,766
VI. Kentucky, Tennessee, Missouri	1,460,103	1,349,592	2,809,695	317,075	320,524	637,599	2,171	2,374	2,269
VII. South Carolina, Georgia, Florida, Alabama	758,544	728,324	1,486,868	682,803	695,662	1,378,465	9,001	9,551	9,270
VIII. Mississippi, Louisiana, Arkansas, Texas	775,983	680,406	1,456,389	547,545	534,412	1,081,957	7,056	7,854	7,429
IX. California, Oregon, Washington, New Mexico, Utah, Dacota, Nevada	403,295	184,272	587,567	3,077	1,402	4,479	76	76	76
Total...............	13,844,028	13,112,934	26,956,962	2,216,738	2,225,990	4,442,728	1,601	1,697	1,647

in the early and middle periods of life. In others, the youthful and middle-aged have emigrated to more promising fields of enterprise, leaving their children and parents behind, and consequently exhibiting the very young and the older ages in relatively larger proportions.

Again, in what may be called immigration districts, it is natural to expect that there should be found a greater proportion of males, at any rate until the cultivation of his land, or the obtaining steady employment, allows the unmarried emigrant to feel that he is settled enough to support a wife and family, or, if he is already married, to bring them over from abroad. On the other hand, more females may be expected in proportion to the total inhabitants where the younger portion of the males emigrate from the district to cultivate new lands, or to push their fortunes in newly-rising localities.

In a table which is given of the growth of population in each district in each decennial period of years since 1790, the numbers somewhat differ in 1860 from Table A, which I have given above, but not so as materially to affect the results; and as the relative growth is more interesting than the actual numbers, the following table will suffice to show the rate per cent. at which each district has increased in the last 70 years:—

B.—*Rate per Cent. of Increase of Population of the United States, in Districts, in each Decennial Period.*

Decennial Period.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
To 1800	34	..	31	..	16	199	53		
„ 1810	33	..	29	432	12	111	31	1,220	
„ 1820	24	..	25	192	10	53	45	107	
„ 1830	27	255	25	83	13	43	48	57	
„ 1840	20	668	25	79	2	31	33	64	
„ 1850	24	191	23	47	16	33	26	120	
„ 1860	20	130	27	44	14	29	17	83	225

This table affords the most singular results. We remark not only the rapid growth of population in the first 10 years after the settlement of some of the districts—as VIII., principally slave States, and IV., which commenced in the same period; II., which increased with two-and-a-half-fold greater rapidity in its second 10 years; and IX., California, &c., which 20 years ago was not included in the census as possessing any inhabitants—but we also see the varying rate of progress of the different States in the last 10 years, from V., containing some of the oldest settled States, which has increased only 14 per cent., to IX., California and the new States, which in the same period has increased 225 per cent. District II. has continued at a very high rate throughout the whole period of 40 years.

The total white population at the census in 1850 was stated to be 19,553,068; in 1860, 26,956,962, an increase of 37·9 per cent. The coloured, including free and slaves, in 1850 was 3,638,808; in 1860, 4,442,728, showing an increase of 22·1 per cent.; and on the total, from 23,191,876 to 31,399,690, the rate of increase was 35·4 per cent.

Population classed under ages.—The next most important inquiry is as to the distribution of the population under ages. All the conditions of age, sex, colour, and periods of settlement, affect the diseases and the mortality of different classes, so that it is necessary to distinguish them. But as the comparison can only be usefully made by reducing the relative proportions to a common standard, say 10,000, which Dr. Jarvis has taken, I have not thought it necessary to give the numbers at each period of age, except for the whole of the United States, distinguishing males and females, and white and coloured populations.

The total number is given as 31,425,480, which differs from the number under districts, 31,399,690, by 28,790, and also in all the districts, and by a much larger number, from the table in which the increase of population is shown, 30,705,290. In this table District IV. alone differs by more than 674,000, and District IX. by nearly 10,000. Table N also, which shows the growth of population in all the nine districts for the last 10 years only, differs from both these tables. There may be reasons for these discrepancies, but they do not appear in the Report.

I have given the table as printed in the Report, to 10,000 of each sex, though I think the better proportion would have been for 10,000 of both sexes, as the ratio of males to females at each age would then be more easily seen. But I have added the actual total numbers from which these relative proportions can be deduced, as well as the actual numbers living at each age and of each sex, by anyone who desires to extend this inquiry and has not the original Report to refer to. (Table C.)

This is a table which allows of several points of comparison of great interest. The district which shows the largest proportion of white children under 15 is VIII. (Mississippi, Louisiana, Arkansas), being 4,471 in 10,000. The one which shows the least, with the exception of IX., is District I., being only in the proportion of

3,446 in 10,000 of the whole population. District IX., California, New Mexico, &c., is wholly exceptional, comprising so many new scarcely-settled territories. Here the proportion of children under 15 is only 2,792 in 10,000 of both sexes, but the number of female children is so greatly in excess of the males, that whilst the former show a proportion of 4,333 the latter are only 2,088 in 10,000. But in this district, in the middle periods of life, the males vastly predominate in numbers, being from 20 to 50—6,925 males to 4,264 females in 10,000 total population of each sex.

In none of the other districts is there any such material difference in the numbers of the sexes at any of the periods of life. Under 15 the largest proportion of female children is in District VIII., 4,666, where the male children under the same age are 4,301 in 10,000; and the smallest proportion of females under 15 is in District I., 3,382, where the males under the same age exceed them in numbers, being 3,517 in 10,000.

The same remark with regard to the District IX. may be made of the coloured population. The females under 15 are 3,105, whilst the male children are only 1,426, in 10,000 of each sex; whilst from 20 to 50 the males exceed the females in the proportion of 7,196 to 5,485 in every 10,000 of each sex. In actual numbers the disproportion is greater still, as there were only 1,402 females to 3,077 males in the whole district.

The total number of white males in the United States was given as 13,844,028, of females 13,112,934, showing an excess of males 731,094, or 105·6 males to 100 females.

In England and Wales only, at the census in 1861, the males were found to be 9,825,246, the females 10,380,258, an excess of 555,012 females, or 94·65 males to 100 females. The total population of Great Britain was then found to be 29,031,164.

The total males of the coloured population of the United States is given as 2,216,738, and females 2,225,990, showing the small excess of 9,252 females.

C.—*Population of the United States in* 1860, *classed under Ages.*

Ages.	White.			Coloured.			Proportion to 10,000 of all Ages.					
							White.			Coloured.		
	Males.	Females.	Persons.	Males.	Females.	Persons.	Males.	Females.	Persons.	Males.	Females.	Persons.
Under 1	409,914	397,527	807,441	61,642	64,818	126,280	296	303	299	278	292	285
1 to 5	1,681,569	1,628,485	3,310,054	293,537	299,267	592,804	1,215	1,242	1,229	1,332	1,352	1,342
5 „ 10	1,788,731	1,739,404	3,528,135	317,999	319,807	637,806	1,293	1,327	1,310	1,443	1,444	1,444
10 „ 15	1,590,785	1,523,314	3,114,099	307,374	294,277	601,651	1,150	1,162	1,156	1,395	1,329	1,362
15 „ 20	1,402,432	1,452,476	2,854,908	245,104	256,490	501,594	1,014	1,108	1,060	1,112	1,158	1,135
20 „ 30	2,512,116	2,421,001	4,933,117	394,185	389,418	783,503	1,816	1,847	1,831	1,789	1,759	1,774
30 „ 40	1,878,302	1,636,525	3,514,827	247,378	253,129	500,507	1,358	1,249	1,305	1,122	1,143	1,133
40 „ 50	1,227,858	1,058,317	2,286,175	162,220	162,299	324,519	887	807	848	736	733	734
50 „ 60	741,348	659,264	1,400,612	93,106	90,588	183,694	536	503	520	422	409	416
60 „ 70	400,706	380,023	780,729	53,906	52,564	106,470	289	290	289	244	237	241
70 „ 80	153,663	156,584	310,247	18,629	19,555	38,184	111	119	115	84	88	86
80 „ 90	38,005	42,753	80,758	5,631	6,903	12,534	27	32	29	25	31	28
90 „ 100	4,135	5,634	9,769	1,644	2,241	3,885	2	4	3	7	10	8
100 and upwds.	385	542	927	799	1,140	1,939	..	..	..	3	5	4
Unknown	14,079	11,085	25,164	13,764	12,394	26,158						
	13,844,028	13,112,934	26,956,962	2,216,738	2,225,990	4,442,728						

C (continued).—*Population in* 1860, *classed under*

Ages.	DISTRICT I.		II.		III.		IV.	
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.
WHITE.								
Under 1	259	258	303	331	304	295	320	336
1 to 5	1,062	1,019	1,310	1,429	1,224	1,204	1,308	1,365
5 „ 10	1,155	1,113	1,294	1,402	1,316	1,296	1,350	1,418
10 „ 15	1,041	992	1,080	1,146	1,161	1,134	1,190	1,218
15 „ 20	994	1,050	944	1,053	1,024	1,096	1,061	1,150
20 „ 30	1,777	1,962	1,748	1,764	1,703	1,830	1,796	1,796
30 „ 40	1,425	1,395	1,466	1,303	1,311	1,265	1,286	1,181
40 „ 50	1,021	939	943	788	918	844	816	743
50 „ 60	652	630	543	454	561	535	500	445
60 „ 70	388	398	262	227	316	320	254	235
70 „ 80	170	184	82	72	124	136	88	84
80 „ 90	44	54	15	14	29	35	19	19
90 „ 100	4	6	1	1	2	4	2	2
100 & upwds.	..	..	..	..	..	..	..	..
Actual Nos.	3,441,151	3,500,919	902,708	805,218	1,750,676	1,745,282	3,176,693	2,949,285
COLOURED.								
Under 1	224	210	291	346	259	240	281	292
1 to 5	897	839	1,148	1,333	1,036	986	1,185	1,180
5 „ 10	1,087	1,022	1,269	1,450	1,238	1,151	1,360	1,396
10 „ 15	1,152	1,023	1,089	1,213	1,255	1,145	1,312	1,304
15 „ 20	992	1,060	884	1,104	1,074	1,128	1,078	1,218
20 „ 30	1,764	1,922	1,930	1,961	1,731	1,931	1,811	1,843
30 „ 40	1,490	1,499	1,607	1,282	1,266	1,312	1,196	1,122
40 „ 50	1,147	1,087	966	732	988	959	791	780
50 „ 60	664	673	521	300	614	571	533	443
60 „ 70	369	385	209	155	333	328	280	246
70 „ 80	152	178	61	76	139	152	110	107
80 „ 90	37	67	9	35	48	61	35	45
90 „ 100	13	20	4	5	10	20	13	10
100 & upwds.	4	6	4	2	2	8	7	6
Actual Nos.	34,897	38,819	4,387	3,930	38,785	43,482	28,894	28,531

Ages, in Districts.—Proportion to 10,000 *of all Ages.*

WHITE.									
V.		VI.		VII.		VIII.		IX.	
Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.
301	289	327	338	316	315	299	327	199	421
1,248	1,212	1,318	1,359	1,290	1,285	1,311	1,425	751	1,575
1,381	1,336	1,407	1,468	1,463	1,471	1,438	1,564	648	1,368
1,269	1,220	1,263	1,300	1,373	1,364	1,253	1,350	490	969
1,066	1,111	1,059	1,166	1,100	1,200	983	1,162	489	903
1,694	1,791	1,837	1,805	1,742	1,768	1,881	1,801	3,232	2,169
1,203	1,189	1,233	1,125	1,113	1,085	1,318	1,136	2,736	1,450
850	829	767	702	750	702	832	664	957	645
529	525	452	408	464	433	429	346	346	309
298	312	226	217	261	244	186	158	113	130
117	130	81	83	93	95	50	48	25	38
30	38	21	22	24	27	10	12	7	12
4	5	2	3	3	5	1	2	1	3
..	..	..	..	..	..	..	..	..	..
1,174,875	1,169,636	1,460,103	1,349,592	758,544	728,324	775,983	680,406	403,295	184,272
COLOURED.									
273	286	313	319	284	295	262	289	130	314
1,319	1,335	1,421	1,436	1,389	1,394	1,283	1,341	549	1,192
1,472	1,465	1,569	1,567	1,463	1,457	1,362	1,392	442	864
1,465	1,379	1,492	1,434	1,420	1,344	1,270	1,236	305	735
1,117	1,131	1,163	1,175	1,116	1,151	1,091	1,197	377	807
1,662	1,632	1,725	1,645	1,760	1,743	1,996	1,954	2,504	2,250
1,047	1,092	1,013	1,059	1,101	1,142	1,241	1,209	2,920	2,128
711	723	622	647	716	729	800	754	1,772	1,107
473	470	385	384	390	389	393	350	760	500
294	293	197	208	238	228	212	189	165	78
113	121	66	78	80	81	58	57	61	14
34	44	20	27	24	27	17	18	9	..
9	14	5	8	7	8	5	5	..	..
3	6	1	4	4	4	3	4	..	7
559,275	558,228	317,075	320,524	682,803	695,662	547,545	534,412	3,077	1,402

Dependent and productive ages.—One of the most important questions connected with the census is to ascertain what part of the population is to be found at ages dependent upon the exertions of others for support, and what part may be considered as adding by their labour or educated skill to the productiveness and wealth of the whole country. It is generally understood that in the United States, amongst the native population, the age at which children begin to be independent of the exertions of their parents is much earlier than in the old countries of Europe. This may be due to a variety of causes, but principally to the wide field opened for enterprise, either in bringing uncultivated lands into use, and the early profits with which a rapidly-increasing population rewards such exertions, or in commercial and industrial pursuits, which for the same reasons raise the value of labour, whether skilled or natural, in the market, and tempt youth earlier into the active pursuits of life.

We shall not be far wrong in assuming that from birth to 15 may be called dependent ages, and though from 15 to 20 may be spent in acquiring the preliminary knowledge of a business or profession, it is only from 20 to 60 that on an average labour or skill will be found to produce an excess so as to contribute to the support of others, such as is implied in the maintenance and education of a growing family. No doubt there are individuals and whole professions and occupations which afford an excess of productive power after the latter age, but for the purpose of inquiry into the averages of a whole nation these limits of age are sufficiently wide for examining the distribution of the population.

D.—*Ratio in* 10,000 *of Population of the United States in* 1860 *under* 20, 20 *to* 60, 60 *to* 80, *and above* 80 *Years of Age.*

District.	WHITE.				COLOURED.			
	Under 20.	20 to 60.	60 to 80.	Over 80.	Under 20.	20 to 60.	60 to 80.	Over 80.
I.	4,468	4,901	570	54	4,248	5,125	544	75
II.	5,133	4,526	322	16	5,044	4,670	251	28
III.	5,027	4,483	448	35	4,750	4,691	476	75
IV.	5,354	4,286	331	21	5,303	4,260	372	58
V.	5,218	4,306	428	38	5,623	3,906	410	56
VI.	5,496	4,168	304	25	5,944	3,739	274	34
VII.	5,558	4,030	347	29	5,658	3,985	313	38
VIII.	5,538	4,221	222	12	5,360	4,349	259	26
IX.	3,412	6,425	148	10	2,462	7,339	184	8
England	4,504	4,759	679	55				
France	3,318	5,623	990	67				
Belgium	4,131	4,973	817	76				
Sweden	4,491	4,723	733	52				

In the older States on the Atlantic coast the proportion under 20 is the smallest, and between 20 and 60 the largest, of all the States, except District IX. Here, as we have constantly remarked, the newness of immigration throws the greatest proportion into the middle and active periods of life. The largest proportion under 20, and the smallest from 20 to 60, is found in Districts VI. and VII., in which likewise the coloured population gives similar results, but to a still greater degree, they being principally slave States. The inference we draw is, that the unhealthiness of climate and other social causes tell in these districts unfavourably upon the constitution at an earlier period of life, leaving a much smaller number to attain the productive and advanced ages of life.

In the comparison added of some of the European populations, it will be seen that England and Sweden, with nearly the same proportion under 20, both show a smaller proportion between 20 and 60 than District I., the old Atlantic States of America, but a much larger proportion in the old-age periods. France, on the other hand, having a much smaller proportion under 20 than either any of the European nations or American States, shows a higher proportion than any between 20 and 60, and a still higher relative proportion at the advanced ages.

In the whole of the United States the proportion of the people under 15 is given as 40·5 per cent., in England 35·4 per cent., in France only 28·7 per cent.; and between 15 and 70, in the United States 58 per cent., in England 61·9 per cent., in France 67·5 per cent.

Military ages.—For the purpose of considering the defensive (we trust the term will always be more appropriate than the aggressive) strength of a nation, the proportion of the male population constantly living between 20 and 40 must be taken, not as implying that all could be abstracted from labour to engage even for a time in military service, since even warriors must at least be fed and clothed, but as showing the usual limits within which the military strength of a nation is confined. It indicates, in comparison with other countries, what are its reserves of force, and its power of sustaining prolonged wars.

In the United States the actual numbers between 20 and 40 were stated to be $\frac{5,030,981}{16,060,666}$, or 3,137 in 10,000 of the whole male population; but this includes the coloured population, of whites only it would be $\frac{4,390,418}{13,844,028}=3,171$ in 10,000.

In England at the same ages, $\frac{4,686,657,}{9,963,425}$ or 4,703 in 10,000 males. In France, $\frac{5,450,551}{17,800,219} = 3,062$ in 10,000 males.

Dr. Jarvis gives the numbers of the male population between 20 and 40 in the late wars of the loyal States as 3,606,147, against 825,400 of the disloyal States; in which I presume he does not count the black population, though they are included in the first estimate of numbers at the military ages.

Foreigners.—One of the most striking peculiarities of the population of America is the large proportion of foreigners to be found at every census. This, in fact, and not the fertility or healthiness of the native population, is the cause of the extraordinarily rapid growth of the nation. In one sense the portion of population which comes from a foreign country may rather be considered as the sign of the growth and rapid increase of the country from which it comes, than of the newly-adopted country, since they spring from the former, and by their secession leave room for the labour and sustenance of a new population, by whom the void caused by emigration is rapidly filled.

Almost every nation in Europe, and Asia too in great part, sends its representatives to the United States. In 1860 no less than 4,136,175 foreigners were found by the census in the different States, and there was scarcely a State which did not contain one at least of the natives of almost every country in Europe. There was no Russian in New Hampshire, no Spaniard in Arkansas, no Belgian and no Austrian in Delaware, no Wirtemberger nor Norwegian in Vermont; but with these exceptions there were some of all classes of foreigners in each State.

Migration.—There is also a constant migration from one State to another, not absolutely caused by want of means of subsistence, but by ambition to rise in life, and take advantage of the constant new openings which may be found in a country which, whether in agriculture, industry, or commerce, presents such incessant movement, progress, and change. In Colorado territory there were but 312 in 100,000 persons born in the territory; 91,040 were born in other States, 77,777 were born in foreign countries, 2 at sea, and 866 unknown. On the other hand, in North Carolina 95,866 in every 100,000 were born in the territory, 3,604 in other States, only 498 in foreign countries, 1 at sea, and 29 unknown.

In the whole of the United States in 1850 the number of the white population living in their native States was 13,540,832, or

76·6 per cent., leaving 23·4 per cent. living out of them. In 1860, 17,527,069, or 75·2 per cent., in their native States, and 24·8 per cent. out of them. In England the proportion living in the counties in which they were born, in 1851 was 75·1, and in 1861 75·4 per cent. But there is a great difference between an English county and an American State. If the counties in each State of America were similarly examined, the proportions of those who had emigrated from one part to another would be found vastly increased.

The migration of the native population is generally from the east to the west, from the older States to the new, as is easily seen by the rate of increase shown in Table B.

Irish emigrants.—It is a singular result of emigration, that both the Irish and the Germans, who form together by far the largest proportion of foreigners seeking a new home in the United States, and who are, in their own country, generally small farmers or of the agricultural class of labourers, seem on their arrival generally to change their occupation, and settle as labourers in commercial and manufacturing cities and towns. This may arise from want of sufficient capital to take and cultivate land, or from the high wages by which they are tempted where manufacturing labour is so scarce. The Irish emigration returns showed that 84·5 per cent. of Irish male emigrants above 15 years of age, and 92·1 per cent. of those under 15, and of the females 79·4 per cent. over 15, and 88·3 per cent. of those under 15, were connected with agriculture. But on examining a table showing the ratio of American, foreign, Irish and Germans, in city populations and in the rest of the State, it is found that in the city of New York the Irish constituted 23 per cent. of the population, but only 12 per cent. of the population in the rest of the State; 16 per cent of the town populations of Philadelphia, Alleghany, Pittsburg and Reading, and only 4 per cent. of the rest of the State.

Foreign immigrants.—The records of foreign emigration are not very complete from 1800 to 1820; but since 1819 the numbers have been officially published, and from that date to 1st June, 1860, they are stated to be—

Males	2,918,619
Females	1,993,162
Sex not stated . . .	48,408
Total . .	4,960,189

Of the foreign arrivals some, of course, proceed through the States to the British possessions in America and other parts; but it is only since 1854 that it has been ascertained that about 2·3 per cent. of the males, and 1·3 per cent. of the females, intended to proceed beyond the United States.

	Males.	Females.	Total.
On 1st June, 1850, the number of foreign emigrants living in the States was found to be . .	1,239,434	1,001,101	2,240,535
On 1st June, 1860, they had increased to	2,225,379	1,906,307	4,131,686

From 1st October, 1819, to 31st May, 1860, upwards of 5,000,000 of foreigners arrived in the United States; and as almost every country in Europe, Asia, and Africa, contribute to swell this multitude, I subjoin the following interesting table of the nativity of the emigrants:—

E., *showing the Nativity of Foreign Emigrants to the United States from* 1819 *to* 1860.

Country.	Number.	Country.	Number.	Country.	Number.
England	302,665	Italy	11,202	Sandwich Islands	79
Ireland	967,366	Belgium	9,862	Cape Verde Islands	29
Scotland	47,890	South America	6,201		
Wales	7,935	Denmark	5,540	Asia	27
Gt. Britain and Ireland	1,425,018	Azores	3,242	Persia	22
		Portugal	2,614	Liberia	19
		Sardinia	2,030	St. Helena	17
Total	2,750,874	Poland	1,659	Iceland	10
		Russia	1,374	Corsica	9
Germany	1,486,044	Central America	968	Society Islands	7
France	208,063	Sicily	560	Morocco	5
British America	117,142	Europe	526	Egypt	4
Prussia	60,432	Madeira Islands	314	Barbary States	4
China	41,443	Canary Islands	286	New Zealand	4
West Indies	40,487	Africa	279	Isle of France	3
Switzerland	37,733	Turkey	170	Cape of Good Hope	2
Norway and Sweden	36,129	East India	127		
		Malta	119	Algiers	2
Holland	21,579	Greece	116	Not stated	180,854
Mexico	17,766	Australia	109		
Spain	16,248	South Sea Islands	79	Total aliens	5,062,414

Dr. Jarvis states the number of emigrants who arrived in the ports of the United States from 1st June, 1850, to 1860, with the intention of remaining, as—

Males	1,526,848
Females	1,107,092
Total . .	2,633,940

In the English Census, 1861, the number of emigrants who left Great Britain in the 10 preceding years is stated to be 2,249,355;

of whom 194,532 were foreigners, and 1,230,986 Irish. But these were not all destined to the United States.

Defects in registration of mortality.—However defective the collection of facts made by the marshals was as to the living population in 1860, it was greater still as to the registration of deaths, which it was intended to perfect for the year ending 1st June, 1860. Mr. De Bow, the Superintendent of the Census for 1850, in his excellent summary of the larger Report, admitted that in that year the schedules were wholly unreliable, and could not furnish the materials for comparing the deaths with the living. Consequently no mortality table could be computed direct from the facts. In 1860 no greater success seems to have attended the effort. The Reports given in by the marshals are deficient on the very face of them. Some even appear to have thought the matter of no consequence as compared with the schedules of the living; and whole counties and towns, both in 1850 and 1860, were returned without any death in the year. Some of the marshals were not intelligent enough to carry out their instructions. Others appear to have trusted to their memories, and recorded the facts at leisure. Even where they were honest, faithful, and painstaking, the difficulties were great enough. Some families might have been dissolved as households by the death of the father or mother, or both, and no one was there to record the family events. If the parents were living, but away, a boarder or visitor, or servant or neighbour, might be called upon to answer, and what accuracy could be expected in reports so obtained? Who, also, would take the trouble to respond for the deaths of strangers, or for those which had taken place in taverns, boarding houses, or on shipboard?

Again, we have the treachery of memory to add to the other liabilities to error. Except in the case of parents answering for their children, the deaths that took place from 3 to 6, 9, or 12 months before might be wholly omitted, or placed in the wrong month, or attributed to the wrong cause. Thus the whole Reports appear to be totally unreliable, so far as it is desired to form an accurate table of mortality from them direct.

The deficiencies vary much in different States. Those in which the largest number of stationary families are found naturally afford their returns most complete. But the mortality differs from 1 in 228 in Washington territory to 1 in 48 in Arkansas. In Massachussets, where a proper registration has been in force for 17 years, the reported deaths do not differ very much from those furnished by the local authorities. But even here it is admitted that in some localities the officials report only such births and deaths as they happen to hear of. Mr. E. B. Elliott, a very able writer on these subjects, thinks that only 166 out of 331 towns, just one-half the number, gave reliable accounts of the mortality. The registers, however, seem increasing in accuracy by degrees. In 1855, 1·84 per cent., in 1860, 1·96 per cent., and in 1863, 2·22 per cent. of the calculated population in that State were returned as deaths, an increase which, it is argued, is not an increase in the actual mortality, but only in the accuracy and care with which the records are kept.

Comparative mortality.—In 1850 the deaths amongst the white and coloured population were distinguished; but in the present census no distinction of race, colour, or nativity has been made; but the causes of death, the sex and age, are recorded for each district. The deaths are accordingly shown as returned, in a series of elaborate tables—the nomenclature of the diseases being reduced in the Census Office to 124 names, and classified under the same heads as Dr. Farr has used in his Reports of the English Census. This system, which is now extensively used on the continent of Europe, was introduced into America by the American Medical Association, in 1846, and forms the basis of classification of diseases for the Census Reports of 1850 and 1860.

Dr. Jarvis concludes that, however defective the actual returns of mortality may be, the omissions may fairly be considered as bearing in most particulars the same proportion as those that are known. Consequently he infers, that, out of a given number of reported deaths, the proportion occurring by certain diseases under or above certain ages in some localities may be compared with similarly-ascertained proportions in other places. Likewise that the proportions in months will allow of a similar safe comparison.

But this manifestly depends on the living population of the compared districts being somewhat alike as to proportions existing at different ages. The diseases which afflict childhood, manhood, or old age, are markedly different. There cannot be a better exemplification of this fact than the records of the Gotha Life Office in Germany, in which the deaths and diseases at all ages have been so carefully analyzed by Mr. Neison, in the *Statistical Journal*, vol. xiii. Unless, therefore, the compared populations have nearly similar proportions of numbers living in childhood, youth, manhood and old age, and also the due ratio of the sexes to each other, an excess of mortality by one class of diseases may depend, not upon an unusual rate of death, but to there being a greater number living liable to such cause of mortality.

We have already, in Table C, seen how widely the proportions living at different ages differ in the enumerated districts, and I will, therefore, only briefly refer the reader, and especially the medical reader, to the comparisons and conclusions which Dr. Jarvis draws from the tables, and proceed to examine the table which he finally gives as the only one showing the actual rate of mortality in the United States of America during the last decennial census period to 1860.

Rate of mortality, 1850 *to* 1860.—The table from which alone, out of all the elaborate and carefully-prepared tables given in the Report, the actual mortality in the whole of the United States during the decennial period to 1860 can be approximated, is not taken direct from the numbers of deaths reported by the marshals, but has been obtained by certain calculations and deductions from the two Censuses of 1850 and 1860. Assuming that the returns of the living are in both cases accurate, that there was no increase of the population by immigration, and no diminution of it by emigration, but only by death, in the two periods, the survivors of the population between each ten ages at one census will at the next decennial census be living between the ages respectively ten years older. The diminution from one period to the other would then be the deaths which have occurred in the interval, and thus an approximate rate of mortality may be obtained for the ten years.

The white populations of the different States have been much influenced by emigration and by migration from one to another during the period in question. As between the North and the South the interchange of the native population is considered to have been so nearly equal, and the compensation so far complete, that, for the purpose of this deduction, each section may be reckoned upon having retained all its own children and received none from the other. Then with regard to the foreign immigrants, a careful analysis was made of those who arrived between 1850 and 1860; their rate of mortality has been computed, and the survivors separated from the total whites at the corresponding ages, thus reducing the numbers by the Census of 1860 to the survivors of those in 1850.

But the Census of 1850 was manifestly incomplete at the early ages of life, as the computations by this process would actually show no deaths amongst those who were under 5 in 1850, and should be between 10 and 15 in 1860. Consequently, under 10 the ages in the table are wholly omitted, and no comparison can be made.

Dr. Jarvis does not give the computations on the ages of the emigrants by which he arrived at the rate of decrement in the ten years, though he gives the final results, showing a mortality of 2·815 per cent. amongst the male emigrants, and 1·453 per cent. amongst the females. With these explanations we must take it on trust, based on his experience in the comparison of facts, and on the earnest fidelity with which he has endeavoured to arrive at the truth. But we should have been glad of some information as to the ages and actual rate of mortality amongst the emigrants in the decennial period, so as to see by what process the number of survivors in 1860 was ascertained. The table used was the English Life Table, checked by further computations by the Irish Life Table.

In regard to the coloured population these computations are unnecessary. None are amongst them, except those who were born in the land, and very few indeed are stated to have left it, so that the comparison of the coloured populations at each census suffices for the table on the principle explained.

By taking the mean age of entering on the decennial period, and dividing Dr. Jarvis' numbers by 10, the average annual mortality will be obtained in a form to compare with the mortality in England for nearly the same period.

F., *showing the Annual Rate of Mortality per Cent. in the White and Coloured Populations of the United States on the Average of the* 10 *Years* 1850 *to* 1860.

Mean Ages.	White.				Coloured.				England, 10 Years 1851 to 1860. Annual Rate of Mortality per Ct.	
	North.		South.		North.		South.			
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.
15–25	·560	·620	1·007	1·375	. .	. .	1·169	1·335	·766	·796
25–35	2·169	2·470	2·077	2·455	1·645	2·143	2·116	2·197	·957	·993
35–45	1·330	1·445	1·883	1·783	1·828	1·777	1·905	2·202	1·248	1·215
45–55	2·202	2·001	2·544	2·556	2·710	2·718	2·687	3·084	1·796	1·520
55–65	2·120	1·866	3·117	2·886	3·541	3·199	2·955	2·908	3·086	2·701
65–75	4·080	3·730	5·014	4·787	5·081	4·416	5·891	5·629	6·533	5·866
75–85	6·480	6·010	6·862	6·513	6·288	5·171	6·505	6·039	14·667	13·434
85–95	8·730	8·400	8·660	8·213	7·118	6·598	7·044	6·396	31·008	28·956
95 & upw^{ds}.	9·100	9·140	8·591	5·789	5·789	5·829	4·696	4·167		

The most remarkable part of this table is, the much smaller decrement in each decade after 75 years of age amongst the coloured, both male and female, and both in the North and the

South, than amongst the whites. Indeed, in the original returns, the proportion of blacks living to whites above 80 is so great, whilst from 70 to 80 the relative number sinks suddenly, that it is manifest the latter ages have been favoured at the expense of the decade after 70. The blacks living at 80 to 90 are given as nearly equal to the whites; but at 90 to 100 the males are 2⅓ times as numerous, and the females 1¾ times; and above 100 they are actually 11 or 12 times the number of the whites.

In the Census Returns, 1860, the whites above 100 are given as 385 males and 542 females, and the coloured with one-sixth of the same population as 799 males and 1,140 females above 100 years of age. No doubt the marshals, in their strict recording of facts stated to them, have, without further inquiry, taken the common phrase of the negroes, "most a 100" or "more than a 100," as literally true. The common tendency of very old people to exaggerate their age expands to still wider dimensions amongst the imaginative blacks.

The table above given is now, as far as the approximation, can be relied on, in the form in which it can be compared with European tables. Dr. Jarvis's table showed, for instance, that of 1,299,299 white males in the Northern States between 20 and 30 in 1850, there were 1,041,191 between 30 and 40 in 1860; the difference or decrement being 258,108, or 19·86 per cent. in ten years on the former number. The rate of decrement given against these ages in his table is 21·69. The actual decrement on the mean of those two numbers living would be 22·06, whence we infer that, with some slight corrections, which have probably been carefully made, the numbers in this table divided by 10 would give the average annual mortality on 100 living at the mean ages between the two periods—that is, at 20, 30, 40, 50, 60, 70, 80, 90, 100, or between 15–25, 25–35, &c. I have accordingly added, from the Registrar-General's 25th Report, the average annual mortality in England for the ten years 1851 to 1860, between the ages 15–25–35, &c., of males and females, above assumed, for comparison with the rate of mortality in America during nearly the same period of years.

It will be remarked, that for the male whites in the North the rate in America is much above the English rate up to the age of 55, whilst above that age it is so very materially less as to give rise to great suspicion of the real truthfulness of the results. For the female white population of the North the same observation applies, but even in a more marked degree, in the higher rate shown at the ages of 25–35; and above the age of 55 the low rate of mortality is still more remarkable when compared with English female life.

Amongst the whites in the South the rate at ages 25–35 in America is somewhat lower, both amongst males and females, than in the North, but above that age considerably higher up to 85; though it still presents, both amongst males and females above 65, the remarkable feature of showing a very much less rate of mortality at those ages than in England.

In the coloured population of the North the rate of mortality amongst females is much less than amongst males in the North at the ages 55 and upwards; but it approximates much more nearly in the South, whilst under 55 it is considerably in excess of the rate amongst the coloured male population of the South. But even in this class, though the rate of mortality is nearly double the rate of the female population of England at the ages below 55, it actually shows considerably less at the ages above 65.

Considering the composition of the American population, and the new conditions under which the laws of vitality prevail there, we might expect some anomalies in and some differences from the European rates of mortality. But these features of the table are too singular not to require more elucidation on another occasion.

Many other important conclusions might be drawn from a minute examination and careful correction of the details furnished in the Report; but I regret to think that the great bulk of the elaborate tables therein given are, in their present form, too defective to afford safe grounds for argument of any kind. The attempt to collect the statistics of mortality even for a single year, by the census marshals, is clearly a failure, and probably under no circumstances would be trustworthy. Perhaps if their inquiry was confined to the deaths within a month, and they were strictly enjoined to obtain the reports from every family or household, they might be sufficiently accurate for that brief period; but the period itself would be too small for an average if converted into the corresponding rates for a year. The only resource is a much more minute and general registration, in all cases compulsory, of births, deaths, and marriages, as to age, sex, colour, and districts, and whether native or foreign, and especially as to emigrants arrived since the last census.

If, then, the marshals at a future census were required to complete the enumeration in a single night, under the present heads of inquiry, the copies of the registration books would afford the means of elucidating some of the most striking problems in population and in political and social economy on which at present we are almost wholly in the dark. The mortality amongst emigrants and in the white and coloured populations, the effects of change of climate and occupation, of the quiet cultivation of land, of fevered life in cities, of the excitement of gold-mining or oil speculations, of early or late marriages, of people fleeing from poverty and misery to competence and sometimes sudden wealth, might all find some light thrown on them by a census so carefully conducted. It would be worthy of the wealth and enlightenment of the American nation to make the attempt, and it is not to be expected but that such a country would triumph over all the difficulties which would naturally attend it. These difficulties, we imagine, will stimulate rather than discourage them in effecting the improvements suggested, and in giving to their census the character which it ought to bear amongst the nations of the world.

On the Construction of Tables by the Method of Differences. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

[Read before the Institute, 25th March, 1867.]

SECTION III.—*On the Construction of Tables in which the Characteristic Function is Rational and Integral.*

(85). We have now to apply the principles laid down in the previous sections to the construction of tables, first of rational, and secondly of irrational, functions. It is only formations of the class last mentioned that are of real importance. A little previous attention to the class first mentioned, however, is desirable, as affording an opportunity of elucidating, under elementary conditions, the arrangement of the work which is found most convenient in practice. The present section will, therefore, be devoted to the formation of successive values of rational functions.

(86). The operation by which successive values of a function are formed, when an initial value and its differences are given, is simply a reverting of that by which, when the requisite number of successive values is given, we thence deduce the differences. The differences being formed by subtraction, the successive values are built up, so to speak, by addition. Thus, take the example in (65), or rather its converse, and let the data be

$$u_x=1438,\quad \Delta u_x=541,\quad \Delta^2u_x=120,\quad \Delta^3u_x=12,\ \text{constant;}$$

then, arranging them in a line, we ascend to successive values of u_x as follows:—

Δ^3	Δ^2	Δ^1	u
12	120	541	1438
	132	661	1979
	144	793	2640
	156	937	3433
	168	1093	4370
	180	1261	5463
	⋮	⋮	⋮

The first column is not filled up, as it would consist of a mere repetition of the value at the top. The second is formed by continuous addition to the initial term of the constant third difference; and the third and fourth are formed in like manner by continuous addition to their initial terms of the successive terms in the preceding columns respectively.

(87). From the manner of formation just described it is easy to show that the nth term in

The first column is Δ^3
second „ $\Delta^2+n\Delta^3$
third „ $\Delta^1+n\Delta^2+\frac{n(n-1)}{2}\Delta^3$
fourth „ $u_x+n\Delta^1+\frac{n(n-1)}{2}\Delta^2+\frac{n(n-1)(n-2)}{2.3}\Delta^3$,

which (70) is equal to u_{x+n}.

(88). The above may be generalized as follows:—If u_x be a function of m dimensions in x, then $\Delta^m u_x$ being constant, (60) we shall have—

$$\begin{aligned}
\Delta^m u_{x+n} &= \Delta^m u_x \\
\Delta^{m-1}u_{x+n} &= \Delta^{m-1}u_x+n\Delta^m u_x \\
\Delta^{m-2}u_{x+n} &= \Delta^{m-2}u_x+n\Delta^{m-1}u_x+\frac{n(n-1)}{2}\Delta^m u_x \\
&\vdots \\
\Delta^1 u_{x+n} &= \Delta^1 u_x+n\Delta^2 u_x+\frac{n(n-1)}{2}\Delta^3 u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^4 u_x+\ldots \\
\Delta^0 u_{x+n}, &
\end{aligned}$$

that is,

$$u_{x+n} = u_x+n\Delta^1 u_x+\frac{n(n-1)}{2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^3 u_x+\ldots$$

or, more generally,

$$\Delta^r u_{x+n}=\Delta^r u_x+n\Delta^{r+1}u_x+\frac{n(n-1)}{2}\Delta^{r+2}u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^{r+3}u_x\ldots$$

the series being continued till the differences of u_x vanish.

We shall have occasion for these results hereafter.

(89). The operation in (86) may obviously be continued to any required extent. In its present form, however, it would soon become very irksome, and it is found expedient to arrange the work in a different manner. The arrangement adopted is influenced, of course, by the number of differences that have to be taken account of; and therefore, in exemplifying it, a commencement will be made with functions of the second degree, ascending, in this section, to those of the third.

(90). *Example* 1.—Given $u_x = 17x^2 + 22x - 15$; it is required to form a series of successive values of u_x when the initial value of x is 100.

The following are the preliminary operations :—

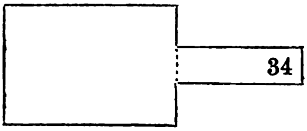
$$\begin{array}{llll@{\qquad\qquad}lll}
17 & 22 & -15(1 & & 17 & 22 & -15(100 \\
 & 39 & 24 & & & 1722 & 172185 = u_{100} \\
 & \underline{56} & & & & & \\
 & 34 & 39(1 & & & 34 & 39(100 \\
 & & 73 & & & & 3439 = \Delta u_{100} \\
 & & \overline{} & & & & \\
 & & 34 = \Delta^2 u_x = \Delta^2 u_{100} & & & &
\end{array}$$

On the left, using the operation in (62), we form the first and second differences of u_x, viz., $\Delta u_x = 34x + 39$, and $\Delta^2 u_x = 34$, constant; and on the right, 100, the initial value of x, is substituted in u_x and Δu_x, giving $u_{100} = 172185$, and $\Delta u_{100} = 3439$. (9). With these elements the required formation is effected, as follows :—

100	172185 3439	110	208105 3779	120	247425 4119
1	175624 3473	1	211884 3813	1	251544 4153
2	179097 3507	2	215697 3847	2	255697 4187
3	182604 3541	3	219544 3881	3	259884 4221
4	186145 3575	4	223425 3915	4	264105 4255
5	189720 3609	5	227340 3949	5	268360 4289
6	193329 3643	6	231289 3983	6	272649 4323
7	196972 3677	7	235272 4017	7	276972 4357
8	200649 3711	8	239289 4051	8	281329 4391
9	204360 3745	9	243340 4085	9	285720 4425
110	208105 3779	120	247425 4119	130	290145 4459

(91). Paper being prepared, and suitably ruled in columns, the successive values of the variable, 100, 101, &c., are filled in as shown. The first difference, 3439, is then inserted on the second line, and the successive terms of this series are formed on the alternate lines by continuous addition of the constant second difference, 34.

(92). This addition is best performed by aid of a card of the form here shown :—



It may be made of thin card or thick cartridge paper; and a slight degree of spring is communicated to the *tongue* by bending it back at the dotted line, to allow the card to be *slidden* down, after each addition, without smearing the newly-written figures. The tongue is pressed down with the fore finger of the left hand, to bring the figures to be added into apposition; and on its withdrawal after the addition is made, the tongue rises from the paper, and the card is slidden down for a new addition. Many applications of the blotting paper are thus saved.

(93). In using the card it is not necessary to form, by actual addition, more than the terminal portion of each term of the series of first differences: the formation in this manner may be restricted to the number of figures that there are on the card. In the present example this number is two; so that the operation with the card is simply the continuous addition of 34, neglecting carriages. We thus produce successively 73, 07, 41, 75, &c. These are then, by the information derived from the first term, 3439, rapidly and unerringly expanded, first into 473, 507, 541, &c., and finally into 3473, 3507, 3541, &c.

(94). It seems hardly necessary to point out how the additions are carried forward from one column to the next. An extra difference is formed, *in pencil*, at the bottom of each column, and is then transferred, in ink, to its place at the top of the next column. See the several columns in the example.

(95). The series of first differences having been completed to the extent required, the first term of the principal series, $u_{100} = 172185$, is inserted in its place, and the remaining terms are formed by continuous addition.

(96). Now as to verification. In consequence of the mutual dependence of the terms, it will suffice to check the several series at intervals, by comparison of a term at each of the points selected for this purpose with the value of the same term formed by an independent process. The agreement of the terms so compared will afford sufficient presumption of the accuracy of the intermediate terms.

(97). If the series to be constructed be of no great extent, the verification values may be formed by Horner's process, (9). The following is the operation for the principal series when the verification values are ten terms apart:—

17	22	$-15(100$
	1722	$172185=u_{100}$
	3422	
17	3422	172185(10
	3592	$208105=u_{110}$
	3762	
17	3762	208105(10
	3932	$247425=u_{120}$
	⋮	⋮

and so on.

(98). But if the table to be formed is of considerable extent, it will be well to have recourse at once, for the construction of the verification values, to an operation analogous to that by which the series to be verified is formed. The operation now in view may be described as follows:—

(99). In differencing an algebraical function we arrive at a constant difference, the *order* of which depends on the dimensions of the function. This has been proved (60), and we have seen it to hold, in the case in which the constant difference of the variable is unit; but it is true also when the difference of the variable is any integer whatsoever. In this case we likewise arrive at a constant difference, and of the same order as in the first case. Hence it appears that we shall be able, by a process analogous to that in (90), to form a series the terms of which shall be equidistant terms of the series to be verified. The common interval between the terms will, of course, depend on the increment given to the variable.

(100). A slight extension of our notation will now be necessary. When the increment of x is unity, the successive differences of u_x are denoted by $\Delta^1 u_x$, $\Delta^2 u_x$, &c.; and when the increment is other than unity, this will be indicated by attaching the increment to the symbol Δ. Thus, if the increment of x be n, the successive differences will be denoted by $\Delta^1_n u_x$, $\Delta^2_n u_x$, &c.; and so also if the increment be 10, that is, if $n=10$, the differences will be denoted by $\Delta^1_{10}u_x$, $\Delta^2_{10}u_x$, &c.

(101). We will now verify the operation in (90) by forming the series whose terms shall be values of u_x corresponding to 100, 110, 120, &c. The requisite differences are formed as follows:—

$$\begin{array}{llll@{\qquad}ll}
17 & 22 & -15(10 & & 340 & 1920(100 \\
 & 192 & 1905 & & & 35920 = \Delta^1_{10}u_{100} \\
 & 362 & & & & \\
\hline
 & 340 & 1920(10 & & & \\
 & & 5320 & & & \\
\cline{3-3}
 & & 3400 = \Delta^2_{10}u_{100} & & &
\end{array}$$

The above operations correspond with the preliminary operations in (90). On the left, changing x into $x+10$, we form $\Delta^1_{10}u_x = 340x + 1920$, and $\Delta^2 u_x = 3400$;* and on the right, by the substitution of 100 for x in $\Delta^1_{10}u_x$, we form $\Delta^1_{10}u_{100} = 35920$. (9). The differences thus formed being used as in (90), we have the operation in the margin, and the terms formed are terms situated at intervals of ten in the series to be verified.

100	172185 35920
10	208105 39320
20	247425 42720
30	290145 46120
40	336265 49520
50	385785 52920
60	438705 56320
70	495025 59720
80	554745 63120
90	617865 66520
200	684385

(102). This series may itself be verified as follows:—

$$\begin{array}{lll}
17 & 22 & -15(100 \\
 & 1722 & 172185 = u_{100} \\
 & 3422 & \\
\hline
17 & 3422 & 172185(100 \\
 & 5122 & 684385 = u_{200} \\
 & \vdots &
\end{array}$$

(103). The series of first differences in the operation of (90) may be verified, ere commencing the final additions, by aid of the formula in (88),

$$\Delta^1 u_{x+n} = \Delta^1 u_x + n\Delta^2 u_x;$$

but it is unnecessary to exhibit the process.

---

* The last step here may well be omitted, since it consists in adding 340 × 10 to 1920, and then subtracting 1920 from the sum. The difference of a function of one dimension is, of course, the product of the first coefficient by the increment of x.

(104). In all cases the verification values ought to be formed first of all, and inserted in their places, so as to avoid interruption to the continuity of the main operations.

(105). *Example* 2.—To form a series of successive values of $u_x = 486298 + 32060x - 12x^2$, the initial value of x being 10.

−12	32060	486298(1	−12	32060	486298(10
	32048	518346		31940	$805698=u_{10}$
	32036				
−24	32048		−24		32048(10
	$-24=\Delta^2 u_x$				$31808=\Delta^1 u_{10}$

10	805698 31808	20	1122698 31568
1	837506 31784	1	1154266 31544
2	869290 31760	2	1185810 31520
3	901050 31736	3	1217330 31496
4	932786 31712	4	1248826 31472
5	964498 31688	5	1280298 31448
6	996186 31664	6	1311746 31424
7	1027850 31640	7	1343170 31400
8	1059490 31616	8	1374570 31376
9	1091106 31592	9	1405946 31352
20	1122698	30	1437298

In this example $\Delta^2 u_x$ is negative, and in consequence the formation of the series of first differences seems to consist of a succession of subtractions of 24. The series is more conveniently formed, however (the last two figures of it, at least), by successive addition of 76, the complement of 24 to a unit of the next higher denomination, and the preceding figures are easily filled in by inspection.

(106). It is worth while to remark that $\Delta^1 u_x$ being in a state of diminution, it will ultimately become negative. At the same point u_x will begin to decrease, and will also ultimately become negative. The first of these changes marks the passage of x through a value which renders u_x a maximum, and the second marks the passage of x through a root of $u_x=0$.

(107). *Example* 3.—To form a series of successive values of $u_x=6x^3+13x^2-22x+17$, the initial value of x being 10.

6	13	−22	17(1
	19	− 3	14
	25	22	
	31		
	18	44	−3(1
		62	59
		80	
		36	62(1
			$36=\Delta^3 u_x$

6	13	−22	17(10
	73	708	$7097=u_{10}$
	18	44	−3(10
		224	$2237=\Delta^1 u_{10}$
		36	62(10
			$422=\Delta^2 u_{10}$

$$\Delta^1 u_x=18x^2+44x-3$$
$$\Delta^2 u_x=36x+62$$
$$\Delta^3 u_x=36$$

(108). The above is the preliminary work for the formation of the differences, &c.; and the following shows the construction of the first thirty terms:—

10	2237	20	8077	30	17517
1	422 2659	1	782 8859	1	1142 18659
2	458 3117	2	818 9677	2	1178 19837
3	494 3611	3	854 10531	3	1214 21051
4	530 4141	4	890 11421	4	1250 22301
5	566 4707	5	926 12347	5	1286 23587
6	602 5309	6	962 13309	6	1322 24909
7	638 5947	7	998 14307	7	1358 26267
8	674 6621	8	1034 15341	8	1394 27661
9	710 7331	9	1070 16411	9	1430 29091
20	746 8077	30	1106 17517	40	1466 30557

10	7097 2237	20	52777 8077	30	173057 17517
1	9334 2659	1	60854 8859	1	190574 18659
2	11993 3117	2	69713 9677	2	209233 19837
3	15110 3611	3	79390 10531	3	229070 21051
4	18721 4141	4	89921 11421	4	250121 22301
5	22862 4707	5	101342 12347	5	272422 23587
6	27569 5309	6	113689 13309	6	296009 24909
7	32878 5947	7	126998 14307	7	320918 26267
8	38825 6621	8	141305 15341	8	347185 27661
9	45446 7331	9	156646 16411	9	374846 29091
20	52777	30	173057	40	403937

The work consists of two portions, of which the second is a repetition of the first (with certain changes), not necessary in practice. On the line corresponding to u_{11}, $\Delta^2u_{10}=422$ is set down; and the succeeding terms of the series of second differences, 458, 494, &c., are formed, *in pencil,* by continuous addition (by aid of a card) of the constant third difference, 36, the last term being $\Delta^2u_{39}=1466$, as may be easily verified by aid of the proper formula of (88). The first difference, $\Delta^1u_{10}=2237$, is now inserted, and by continuous addition, *in ink,* the entire series of first differences is formed, down to $\Delta^1u_{40}=30557$, which may be verified as before. The pencilled second differences, for which there is no further occasion, are now to be rubbed out,* and $u_{10}=7097$ being inserted in its place, u_{11}, u_{12}, &c., are formed in succession by continuous addition. The completed operation is shown on the right, it being understood that in practice this is all that is exhibited. It is formed *upon* the first skeleton, the successive values of u_x occupying the lines left vacant by the erasure of the second differences.

(109). The operation may be verified by the formation of selected terms, either in series, in the manner of (101), or by Horner's process, thus :—

$$\begin{array}{cccc} 6 & 13 & -22 & 17(40 \\ & 253 & 10098 & 403937=u_{40} \end{array}$$

(110). *Example* 4.—Given four consecutive values of a function of the third degree, as below. It is required to continue the series.

It will be perceived that this example differs from those that precede in that here the data are the requisite number of consecutive values, and the form of the function is not given.

	Δ^1+	Δ^2-	Δ^3+
$u_0=3289586$	289674	9246	18
$u_1=3579260$	280428	9228	
$u_2=3859688$	271200		
$u_3=4130888$			

The elements for the construction then are—

$$\begin{aligned} u_0 &= 3289586 \\ \Delta^1u_0 &= 289674 \\ \Delta^2u_0 &= -9246 \\ \Delta^3u_0 &= 18 \end{aligned}$$

in regard to which it is to be observed that Δ^2u_0 is negative. This gives rise, of course, to a modification in the working. The formation of the first twenty terms is as follows:—

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* I may mention here that I find the best way of removing extensive pencil writing is to rub it, both before and after using the India-rubber, with a piece of flannel or other woollen cloth. The first application of the woollen, supposing the pencil used to be a *good* one, leaves little for the India-rubber to do, and the second most effectually removes the *greasy* feel that always remains more or less after the use of India-rubber, and restores the paper to its pristine state.

0	289674	10	198024	0	3289586 289674	10	5772416 198024
1	9246 280428	1	9066 188958	1	3579260 280428	1	5970440 188958
2	9228 271200	2	9048 179910	2	3859688 271200	2	6159398 179910
3	9210 261990	3	9030 170880	3	4130888 261990	3	6339308 170880
4	9192 252798	4	9012 161868	4	4392878 252798	4	6510188 161868
5	9174 243624	5	8994 152874	5	4645676 243624	5	6672056 152874
6	9156 234468	6	8976 143898	6	4889300 234468	6	6824930 143898
7	9138 225330	7	8958 134940	7	5123768 225330	7	6968828 134940
8	9120 216210	8	8940 126000	8	5349098 216210	8	7103768 126000
9	9102 207108	9	8922 117078	9	5565308 207108	9	7229768 117078
10	9084 198024	20	8904 108174	10	5772416	20	7346846

(111). Here $\Delta^2u_0=-9246$ is first inserted, *in pencil*, in the place to be finally occupied by u_1, and the succeeding second differences may then be formed by the continuous *subtraction* of $\Delta^3u_0=18$, Δ^2u_0 and Δ^3u_0 having opposite signs. But a preferable mode of formation is by the continuous addition, by means of a card (92), of 82, the complement of 18 to 100, attending only to the last two figures, 28, 10, 92, &c. The preceding figures can then be filled in by inspection. And verification is obtained by (88), thus:—

$$\begin{aligned}\Delta^2u_{19}&=\Delta^2u_0+19\Delta^3u_0\\&=-9246+19\times18=-8904.\end{aligned}$$

The first difference of u_0, 289674, is now inserted in its place, and, the signs being again opposite, the remaining terms of this series would seem to require to be formed by subtraction. This inconvenience (for such most, if not all, computers would feel it) may, however, be avoided, as follows:—Form, by the proper formula of (88), a remote term of the series, say Δ^1u_{20}, thus:—

$$\begin{aligned}\Delta^1u_{20}&=\Delta^1u_0+20\Delta^2u_0+190\Delta^3u_0\\&=289674-20\times9246+3420\times18\\&=108174.\end{aligned}$$

This term will fall to be inserted immediately under $\Delta^2 u_{19}=8904$; and, *commencing at the bottom*, the first differences may now be formed by addition. After a very little practice it will be found nearly as easy to add *upwards* as *downwards*. It is convenient in this operation to place a piece of blotting paper under the hand, moving it upwards after the formation of every two or three terms. The pencilled second differences are now to be erased, and $u_0=3289586$ being inserted in its place, the required formation is completed by continuous addition. Verification is obtained as follows (88):—

$$\begin{aligned} u_{20} &= u_0+20\Delta^1 u_0+190\Delta^2 u_0+1140\Delta^3 u_0 \\ &= 3289586+20\times 289674-190\times 9246+1140\times 18 \\ &= 7346846. \end{aligned}$$

As in Example 3, I repeat the scheme of the operation, to show the appearance it assumes when the work is finished.

(112). I have now gone as far as is necessary in regard to the construction of rational functions. I have entered on the subject only on account of the opportunity it affords for illustrating the arrangement of the materials operated upon which is found most conducive to facility and perspicuity; and I have gone as far as three orders of differences, which are as many as can be conveniently dealt with in a single column. The mode of dealing with four and five orders will receive ample illustration in the concluding section, which will be occupied with the subject of irrational functions.

(113). To avoid misconception, I desire here to say, that in the actual construction of tables of rational functions it will seldom, if ever, be necessary to have recourse to the formal methods I have exemplified. By careful investigation relations will generally be found to subsist, which will enable the desired construction to be effected by the expenditure of much less time and labour than would be requisite if the methods in question were made use of. Illustration of this remark is afforded by Mr. Laundy's *Table of Quarter Squares*, the methods employed in the construction of which are described by the author in his Introduction, pp. ix. to xiv.

(114). Further illustration is afforded by the specimen hereto appended, which I extract from the *Mechanics' Magazine*, vol. xxii., p. 104. It is one of two pages there given as a specimen of a Table of Squares, by Mr. David Booth, who dates his communication, "Charlotte Street, Bloomsbury, Nov. 5, 1834." It is well deserving of further publicity, on account of the ingenuity of its

Roots.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	Term.
3600	1 296	18 496	55 696	112 896	190 096	287 296	404 496	541 696	698 896	876 096	0000
01	296	498	700	902	104	306	508	710	912	114	7201
02	297	501	705	909	113	317	521	725	929	133	4404
03	298	504	710	916	122	328	534	740	946	152	1609
04	298	506	714	922	130	338	546	754	962	170	8816
05	299	509	719	929	139	349	559	769	979	189	6025
06	300	512	724	936	148	360	572	784	996	208	3236
07	301	515	729	943	157	371	585	799	699 013	227	0449
08	301	517	733	949	165	381	597	813	029	245	7664
09	302	520	738	956	174	392	610	828	046	264	4881
3610	1 303	18 523	55 743	112 963	190 183	287 403	404 623	541 843	699 063	876 283	2100
11	303	525	747	969	191	413	635	857	079	301	9321
12	304	528	752	976	200	424	648	872	096	320	6544
13	305	531	757	983	209	435	661	887	113	339	3769
14	306	534	762	990	218	446	674	902	130	358	0996
15	306	536	766	996	226	456	686	916	146	376	8225
16	307	539	771	113 003	235	467	699	931	163	395	5456
17	308	542	776	010	244	478	712	946	180	414	2689
18	308	544	780	016	252	488	724	960	196	432	9924
19	309	547	785	023	261	499	737	975	213	451	7161
3620	1 310	18 550	55 790	113 030	190 270	287 510	404 750	541 990	699 230	876 470	4400
21	311	553	795	037	279	521	763	542 005	247	489	1641
22	311	555	799	043	287	531	775	019	263	507	8884
23	312	558	804	050	296	542	788	034	280	526	6129
24	313	561	809	057	305	553	801	049	297	545	3376
25	314	564	814	064	314	564	814	064	314	564	0625
26	314	566	818	070	322	574	826	078	330	582	7876
27	315	569	823	077	331	585	839	093	347	601	5129
28	316	572	828	084	340	596	852	108	364	620	2384
29	316	574	832	090	348	606	864	122	380	638	9641
3630	1 317	18 577	55 837	113 097	190 357	287 617	404 877	542 137	699 397	876 657	6900
31	318	580	842	104	366	628	890	152	414	676	4161
32	319	583	847	111	375	639	903	167	431	695	1424
33	319	585	851	117	383	649	915	181	447	713	8689
34	320	588	856	124	392	660	928	196	464	732	5956
35	321	591	861	131	401	671	941	211	481	751	3225
36	322	594	866	138	410	682	954	226	498	770	0496
37	322	596	870	144	418	692	966	240	514	788	7769
38	323	599	875	151	427	703	979	255	531	807	5044
39	324	602	880	158	436	714	992	270	548	826	2321
3640	1 324	18 604	55 884	113 164	190 444	287 724	405 004	542 284	699 564	876 844	9600
41	325	607	889	171	453	735	017	299	581	863	6881
42	326	610	894	178	462	746	030	314	598	882	4164
43	327	613	899	185	471	757	043	329	615	901	1449
44	327	615	903	191	479	767	055	343	631	919	8736
45	328	618	908	198	488	778	068	358	648	938	6025
46	329	621	913	205	497	789	081	373	665	957	3316
47	330	624	918	212	506	800	094	388	682	976	0609
48	330	626	922	218	514	810	106	402	698	994	7904
49	331	629	927	225	523	821	119	417	715	877 013	5201

arrangement, and the facility thereby afforded for the construction of the table, and its continuation to any extent that may be desired.

(115). The page given contains 500 squares. The last column, headed *Terms.* (contraction for *Terminations*), contains on each line the four figures which are the common termination of all the squares upon that line. With this understanding the column headed 0 contains the fifty squares $(3600)^2$ to $(3649)^2$, the column headed 1 contains the squares $(13600)^2$ to $(13649)^2$, the column headed 2 the squares $(23600)^2$ to $(23649)^2$, and so on.

(116). Now as to the mode of construction, which is conducted in the present form. The first difference of x^2 being $2x+1$, when x is 3600 this is 7201; and the constant second difference being 2, the terminal figures of column 0 are formed by continuous addition to 0000, the terminal figures of $(3600)^2$, of the terms of the series 7201, 7203, 7205, &c. (neglecting carriages). The series thus formed, occupying the last column, each term of it belongs equally to all the squares on the same line; for, the roots of each two adjoining squares differing from each other by 10,000, it is obvious that x^2 and $(x+10,000)^2$ have the last four figures common. Now mark the progression in the vertical columns. In column 0 the increase is by 0 and 1, according as there is an increase or a decrease in the leading figures of the last column. In column 1 the increase is by 2 and 3, subject to the same condition; in column 2 the increase is by 4 and 5; and so on up to column 9, in which the increase is by 18 and 19.

(117). It thus appears that a table of squares admits of being formed in this manner with great facility. It would, however, require verification. This would be obtained by addition of the several columns, the expression for the sum of a number of consecutive squares being well known.

On the Value of Annuities payable Half-yearly, Quarterly, &c. By THOMAS BOND SPRAGUE, M.A., *Actuary of the Equity and Law Life Assurance Society.* (*Part III.*)

IN order to complete the consideration of this question as far as it relates to a single life, I will now give some numerical examples showing the magnitude of the corrections, or the additions to the value of annuities when payable half-yearly, as found from the formulæ of Mr. Baily and Mr. Woolhouse. Those formulæ will also be exhibited under some new aspects.

I will commence with the following demonstration of Baily's formula. The difference between the values of an annuity, when payable yearly and half-yearly respectively, consists of two elements:—the first being the interest gained on the sums paid in the middle of each year in which the life continues in being, instead of at the end thereof; and the second being the value of the chance of receiving an extra payment in the middle of the year in which the life drops, in consequence of the life dropping in the second and not in the first half. The annuity being, as usual, supposed to be £1, when the annuity is payable half-yearly there is ·5 paid in the middle instead of at the end of each year which the life survives; and this payment at the end of the year amounts to $\cdot 5\sqrt{1+i}$, and the excess of the value of this over the same payment made at the end of the year is $\frac{\sqrt{1+i}-1}{2}$. The value of all such differences, supposed to be payable at the end of each year which the life survives, is therefore $a_x \times \frac{\sqrt{1+i}-1}{2}$. The second element in the difference is the chance of receiving an extra ·5 in the middle of the year in which the life drops. Now the deaths being supposed to be distributed uniformly, the chance of the life dropping in the second half of the year is exactly equal to the chance of its dropping in the first half: hence the value of this element is one-half that of ·5 to be received in the middle of the year in which the life drops, *i.e.* $= \frac{1}{2} \cdot \frac{\sqrt{1+i}}{2} \cdot A_x$.

Hence the difference between the values of an annuity of £1 when payable yearly and half-yearly is

$$\frac{\sqrt{1+i}-1}{2} \cdot a_x + \frac{\sqrt{1+i}}{4} \cdot A_x.$$

It is easily shown that this agrees with Baily's formula; for since $A_x = \frac{1-ia_x}{1+i}$, the above becomes

$$\frac{\sqrt{1+i}-1}{2} a_x + \frac{1-ia_x}{4\sqrt{1+i}}$$

$$= \left(2\sqrt{1+i}-2-\frac{i}{\sqrt{1+i}}\right)\frac{a_x}{4} + \frac{1}{4\sqrt{1+i}}$$

$$= \left(\frac{2+i}{\sqrt{1+i}}-2\right)\frac{a_x}{4} + \frac{1}{4\sqrt{1+i}}$$

$$=\left(\sqrt{1+i}+\frac{1}{\sqrt{1+i}}-2\right)\frac{a_x}{4}+\frac{1}{4\sqrt{1+i}}$$

$$=\left(\sqrt{v}+\frac{1}{\sqrt{v}}-2\right)\frac{a_x}{4}+\frac{\sqrt{v}}{4}.$$

Adding a_x to this, we get Baily's formula for the value of an annuity payable half-yearly.

In the same way, the excess of the value of an annuity payable quarterly over one payable yearly, or the correction for an annuity payable quarterly, is seen to be

$$\{(1+i)^{\frac{3}{4}}+(1+i)^{\frac{1}{2}}+(1+i)^{\frac{1}{4}}-3\}\frac{a_x}{4}$$

$$+\left\{\frac{3}{4}\cdot\frac{(1+i)^{\frac{3}{4}}}{4}+\frac{2}{4}\frac{(1+i)^{\frac{1}{2}}}{4}+\frac{1}{4}\frac{(1+i)^{\frac{1}{4}}}{4}\right\}\mathrm{A}_x,$$

since if death occurs in the fourth quarter of the year, of which the chance is $\frac{1}{4}$, there is payable at the end of each of the first three quarters, £$\frac{1}{4}$, which payments together amount at the end of the year to

$$\frac{1}{4}\{(1+i)^{\frac{3}{4}}+(1+i)^{\frac{1}{2}}+(1+i)^{\frac{1}{4}}\} \quad . \quad . \quad . \quad . \quad (1)$$

So, if death occurs in the third or in the second quarters, the chance of each of which is also $\frac{1}{4}$, there are payable sums which respectively amount at the end of the year to

$$\frac{1}{4}\{(1+i)^{\frac{3}{4}}+(1+i)^{\frac{1}{2}}\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{1}{4}\cdot(1+i)^{\frac{3}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying each of these quantities (1), (2), (3), by the chance of receiving it ($\frac{1}{4}$), we get the value of the correction for the quarterly annuity as stated above. This may, as before, be shown to be equivalent to Baily's formula, for it becomes

$$\{(1+i)^{\frac{3}{4}}+(1+i)^{\frac{1}{2}}+(1+i)^{\frac{1}{4}}-3\}\frac{a_x}{4}$$

$$+\{3(1+i)^{\frac{3}{4}}+2(1+i)^{\frac{1}{2}}+(1+i)^{\frac{1}{4}}\}\frac{1-ia_x}{16(1+i)}$$

$$=\{4(1+i)(1+i)^{\frac{3}{4}}+4(1+i)(1+i)^{\frac{1}{2}}+4(1+i)(1+i)^{\frac{1}{4}}-12-12i$$

$$-3i(1+i)^{\frac{3}{4}}-2i(1+i)^{\frac{1}{2}}-i(1+i)^{\frac{1}{4}}\}\frac{a_x}{16(1+i)}$$

$$=\{(4+i)(1+i)^{\frac{3}{4}}+(4+2i)(1+i)^{\frac{1}{2}}+(4+3i)(1+i)^{\frac{1}{4}}-12-12i\}\frac{a_x}{16(1+i)}$$

$$+\{3(1+i)^{\frac{1}{2}}+2(1+i)^{\frac{1}{4}}+1\}\frac{v^{\frac{3}{4}}}{16}$$

$$=\frac{v^{\frac{3}{4}}}{16}\left\{(4+i)(1+i)^{\frac{1}{2}}+(4+2i)(1+i)^{\frac{1}{4}}+4+3i-12(1+i)^{\frac{3}{4}}\right\}a_x$$
$$+\left\{3(1+i)^{\frac{1}{2}}+2(1+i)^{\frac{1}{4}}+1\right\}\frac{v^{\frac{3}{4}}}{16}.$$

Adding a_x to this, we get Baily's formula for the value of an annuity payable quarterly.

I have now to draw the reader's attention to the following table, in which are exhibited the values of the correction for a half-yearly annuity, as calculated by the formulæ of Baily and Woolhouse, for the three rates of interest, 3, 4, and 5 per cent., according to the "Experience" table of mortality.

Table showing the Addition to the Value of an Annuity when payable Half-yearly, according to the "Experience" Table of Mortality.

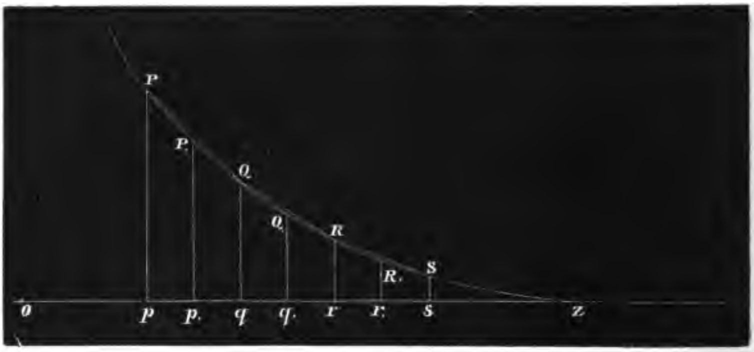
3 PER CENT.			
Age.	B.	W.	W–B.
20	·24752	·24770	·00018
30	·24741	·24763	·00022
40	·24726	·24751	·00025
50	·24709	·24718	·00009
60	·24689	·24630	–·00059
70	·24670	·24412	–·00258
80	·24654	·23909	–·00745
90	·24640	·22512	–·02128
4 PER CENT.			
Age.	B.	W.	W–B.
20	·24692	·24710	·00018
30	·24678	·24703	·00025
40	·24660	·24691	·00031
50	·24634	·24657	·00023
60	·24605	·24569	–·00036
70	·24575	·24352	–·00223
80	·24550	·23849	–·00701
90	·24529	·22452	–·02077
5 PER CENT.			
Age.	B.	W.	W–B.
20	·24634	·24650	·00016
30	·24619	·24643	·00023
40	·24598	·24631	·00033
50	·24566	·24597	·00031
60	·24528	·24508	–·00020
70	·24487	·24292	–·00195
80	·24450	·23789	–·00661
90	·24419	·22392	–·02027

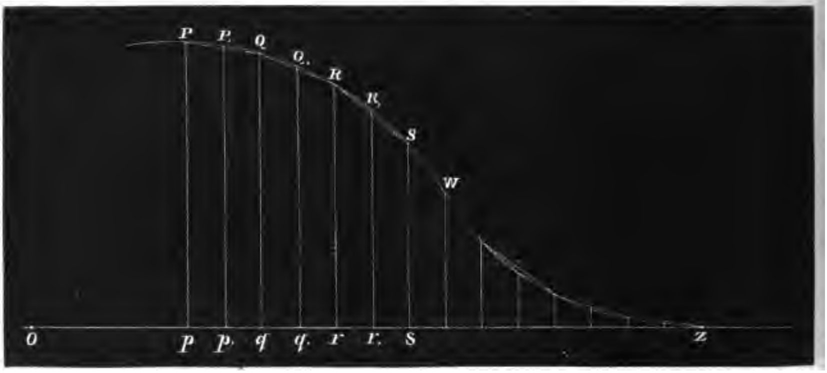
Here the value of the correction as found from Mr. Baily's formula is put under the heading B, and as found from Mr. Woolhouse's formula under W; the difference, or the error in Baily's approximation, being put in the last column under the heading W—B. The formulæ used are $B_t=\left(\sqrt{v}+\frac{1}{\sqrt{v}}-2\right)\frac{a_t}{4}+\frac{\sqrt{v}}{4}$; and $W_t=\frac{1}{4}-\frac{\mu_t+\delta}{16}$.

We might have foreseen that Baily's formula would give the values of the correction too large for ages over 70. For it assumes that the chance of dying in the second half of the year is equal to the chance of dying in the first half, each being ½. But from the age of 72 upwards, the number dying in a year according to the Experience table of mortality continually decreases; so that when the number dying in a portion of a year is calculated more accurately than by constant first differences, we shall have the number dying in the first half of each year after the age of 72, greater than the number dying in the second half. It will follow that the chance of dying in the second half is less than ½, or the chance of receiving the extra £½ is less than Baily's formula supposes. For ages below 72, the contrary is the case; or the number dying in the first half of each year is less than the number dying in the second half; and the chance of receiving the extra £½ is greater than ½. As we consider successively younger ages, the influence of the latter portion of the table of mortality is felt less and less; until we reach an age between 50 and 60, when it is neutralized by the opposite influence of the portion of the table relating to the ages under 72. For that age, Baily's formula will be accurately true. We see also that in the case of a temporary annuity which does not relate to any age over 72, the value as formed by Baily's formula will always be too small.

The values above given as deduced from Mr. Woolhouse's formula may be trusted, as I have shown already (p. 211), to the fifth decimal place—at all events, for ages not exceeding 80. From a consideration of these values we conclude that the common approximation, ·25, is in no case correct to three decimal places, and at advanced ages is not true even in the second decimal place. We also see that Baily's formula, deduced upon the supposition of uniform decrements, gives values scarcely ever correct to more than three decimal places, and at advanced ages is almost as erroneous as the common approximation.

We thus see what a great advance Mr. Woolhouse's simple formula is on all that has been previously written on the subject. Of so much importance does that formula appear to me, that I will endeavour still further to elucidate its meaning before passing on to the consideration of the other problems suggested at the close of the second part of this paper (p. 228).





Let a curve P Q R S . . . . Z be drawn, of which the abscissa represents the age (x), and the ordinate represents $v^x l_x$, (*i.e.* D_x, according to the usual notation).

Let $Op=k$, $pq=qr=rs= \dots =1$, and let pq, qr, . . . be bisected in the points p_1, q_1, . . . Then

$$Pp=D_k,\quad Qq=D_{k+1},\quad Rr=D_{k+2},\ \&c.$$

Also $$P_1 p_1=D_{k+\frac{1}{2}},\quad Q_1 q_1=D_{k+\frac{3}{2}},\ \&c.$$

The value of the annuity at age k is equal to $\dfrac{D_{k+1}+D_{k+2}+\dots\dots}{D_k}$

or,
$$a_k=\frac{Qq+Rr+\dots}{Pp}$$

But since $pq=1$, the area of the quadrilateral $PpQq$ is equal to

$$\frac{Pp+Qq}{2},$$

and that of the polygon $PQR\dots Zp$ is equal to

$$\frac{Pp+Qq}{2}+\frac{Qq+Rr}{2}+\frac{Rr+Ss}{2}+\dots$$
$$=\frac{Pp}{2}+Qq+Rr+\dots$$

whence we see that

$$a_k=\frac{\text{polygon } PQR\dots Zp}{Pp}-\tfrac{1}{2}.$$

Again, the value of the annuity payable half-yearly is equal to

$$\frac{l_{k+\frac{1}{2}}v^{k+\frac{1}{2}}+l_{k+1}v^{k+1}+l_{k+\frac{3}{2}}v^{k+\frac{3}{2}}+\dots}{2l_kv^k}$$
$$=\frac{P_1p_1+Qq+Q_1q_1+Rr+\dots}{2Pp}$$

But the area of the quadrilateral Ppp_1P_1 is equal to $pp_1 \times \frac{Pp+P_1p_1}{2}=\frac{Pp+P_1p_1}{4}$, and that of the polygon $PP_1QQ_1R\dots Zp$ is $\frac{Pp+P_1p_1}{4}+\frac{P_1p_1+Qq}{4}+\dots$

$$=\frac{Pp}{4}+\frac{P_1p_1+Qq+Q_1q_1+\dots}{2}$$

whence the value of the annuity payable half-yearly is

$$a_k^{(2)}=\frac{\text{polygon } PP_1QQ_1\dots Zp}{Pp}-\tfrac{1}{4}.$$

Comparing these expressions for a_k and $a_k^{(2)}$, we see that the common approximation of adding ·25 for the value of the half-yearly annuity is equivalent to assuming that the polygon $PQR\dots Zp$ is equal to $PP_1QQ_1\dots Zp$; *i.e.* the polygon $PQR\dots Zp$ is substituted for the curvilinear area.

The problem of finding the value of an annuity payable half-yearly, quarterly, &c., is thus seen to be essentially one of interpolation. The ordinates Pp, Qq, Rr .... are known quantities; (*i.e.* the values of D_k, D_{k+1}, D_{k+2} .... are tabulated); but the intermediate ordinates, P_1p_1, Q_1q_1, ... (and the intermediate values $D_{k+\frac{1}{2}}$, $D_{k+\frac{3}{2}}$, &c.) are not known, and can only be determined by some process of interpolation, more or less accurate. The common approximation by which ·25 is added to the value of the annuity, is equivalent to supposing that $P_1p_1=\frac{Pp+Qq}{2}$, $Q_1q_1=\frac{Qq+Rr}{2}$, &c., or substituting the chords PQ, QR, ... for the curvilinear arcs. The various other methods of approximation, —those, for example, by which the formulæ of Mr. Baily and Mr. Woolhouse are obtained, and that employed by myself in the second part of this paper (p. 202),—are all equivalent to substituting for the arcs PQ, QR, RS .... some other curve lines whose equations are known, and the ordinates of which can be calculated with any required degree of accuracy.

According to Baily's method, we suppose the number alive at the age $k+x$, where x is less than unity, to be

$$l_k-x(l_k-l_{k+1})=(1-x)l_k+xl_{k+1};$$

so that

$$D_{k+x}=v^{k+x}l_{k+x}=v^{k+x}\{(1-x)l_k+xl_{k+1}\}.$$

The curve indicated by this equation passes through the points P and Q; but does not otherwise coincide with PQ. It may be shown to be always convex to the axis of abscissæ, or to lie below the straight line PQ. For the ordinate of the straight line corresponding to the age $k+x$ is

$$(1-x)v^kl_k+xv^{k+1}l_{k+1},$$

and the curve fixed by Baily's method will lie above or below the straight line according as

$$v^{k+x}\{(1-x)l_k+xl_{k+1}\}> \text{ or } <(1-x)v^kl_k+xv^{k+1}l_{k+1}$$

as $$v^x\{(1-x)l_k+xl_{k+1}\}> \text{ or } <(1-x)l_k+xvl_{k+1}$$

as $$(v^x-v)xl_{k+1}> \text{ or } <(1-v^x)(1-x)l_k$$

$$> \text{ or } <(1-v^x)(1-x)(l_{k+1}+d_k)$$

as $$(1-x-v^x+xv)l_{k+1}+(1-v^x)(1-x)d_k< \text{ or } >0$$

Now $$v^x=e^{-\delta x}=1-\delta x+\frac{\delta^2x^2}{2}-\frac{\delta^3x^3}{6}+\dots.$$

$$v=1-\delta+\frac{\delta^2}{2}-\frac{\delta^3}{6}+\dots.$$

whence $$1-v^x=\delta x-\frac{\delta^2x^2}{2}+\frac{\delta^3x^3}{6}-\ldots.$$

$$x(1-v)=\delta x-\frac{\delta^2x}{2}+\frac{\delta^3x}{6}-\ldots.$$

$$\therefore\ 1-x-v^x+xv=\frac{\delta^2x}{2}(1-x)-\frac{\delta^3x}{6}(1-x^2)+\ldots.$$

$$=\frac{\delta^2x}{2}(1-x)\left\{1-\frac{\delta}{3}(1+x)+\ldots.\right\}$$

which is always positive, since x and δ are both less than unity. Also $(1-v^x)(1-x)$ is always positive.

$$\therefore (1-x-v^x+xv)l_{x+1}+(1-v^x)(1-x)d_x>0,$$

and it follows that Baily's curve lies always below the straight line PQ, or is convex to the axis of x.

The figures given above show two forms which the curve may assume. In the first, the curve is throughout convex to the axis. In the second, the curve is originally concave to the axis, and there is a point of contrary flexure, W, where the curve becomes convex to the axis.

When money is supposed to bear no interest the curve will have the second of the above forms, the point of flexure occurring in the Experience table, at the age 72, when the number dying in a year is a maximum. In that table, the number dying in a year steadily increases from the age of 15 to that of 72; and from the latter age, steadily decreases. In the Carlisle table, on the contrary, the number dying in a year increases and decreases several times; and to each maximum and minimum corresponds a point of contrary flexure in the curve.

But this will not necessarily be the case when we introduce the element of interest. In order that the curve may have a point of contrary flexure, we must have the second difference of v^xl_x change sign. Take the differences of v^xl_x, as follows:—

$$\begin{array}{l|l|l} v^{x-1}l_{x-1} & & \\ & v^xl_x-v^{x-1}l_{x-1} & \\ v^xl_x & & v^{x+1}l_{x+1}-2v^xl_x+v^{x-1}l_{x-1} \\ & v^{x+1}l_{x+1}-v^xl_x & \\ v^{x+1}l_{x+1} & & \end{array}$$

In order that the second difference may change sign, we must have $l_{x+1}-2(1+i)l_x+(1+i)^2l_{x-1}$ change sign, *i.e.*,

$$l_x-d_x-2(1+i)l_x+(1+i)^2(l_x+d_{x-1}),$$

or $$i^2l_x-d_x+(1+i)^2d_{x-1},$$

or $$i^2l_x-(d_x-d_{x-1})+2id_{x-1}.$$

So long as $d_x < d_{x-1}$ this will always be positive, and cannot change sign. The first condition, therefore, necessary for a change of sign, is that d_x shall be $> d_{x-1}$, or that the number dying in a year shall increase as we take higher ages. Hence in the Experience table, the point of flexure, if there is any, will be at an age less than 72. Now in the Experience table, $d_x - d_{x-1}$ is a maximum for $x=62$. At that age

$$d_x - d_{x-1} = 74, \quad l_{x-1} = 54{,}275, \quad d_{x-1} = 1770.$$

If now we take a low rate of interest, say 1 per cent., so that $i = \cdot 01$, we have

$$i^2 l_{x-1} + 2id_{x-1} = 5\cdot 4275 + 35\cdot 40 = 40\cdot 8275,$$

and is therefore less than $d_x - d_{x-1}$. In this case, then, the function changes sign, and there is a point of contrary flexure at some age between 62 and 72. But if we take a higher rate of interest, as 2 per cent.,

$$i^2 l_{x-1} + 2id_{x+1} = 21\cdot 7100 + 70\cdot 8 = 92\cdot 51,$$

and is now greater than $d_x - d_{x-1}$. In this case we may therefore expect, although the reasoning is not quite conclusive, that there is no point of contrary flexure. In fact, an examination of the "D column" as given by Jenkin Jones for the three rates of interest 2½, 3, and 3½ per cent., shows that to be the case, inasmuch as $D_{x-1} - D_x$ continually diminishes as x is increased.

In the Carlisle 3 per cent. table, on the contrary, $D_{x-1} - D_x$ is a maximum at the ages 73 and 61; and a minimum at the ages 71 and 58; and there is therefore at each of those ages a point of contrary flexure.

In the demonstration of the formula for the value of a half-yearly annuity given by Mr. Woolhouse (vol. xi. p. 320) and by myself (vol. xiii. p. 208) it is assumed that the curve touches the axis of x at the point Z. This is implied in Mr. Woolhouse's assumption that $\left(\frac{dV}{dt}\right)_n = 0$; and in mine that the values of $\frac{du}{dx}$, $\frac{d^3u}{dx^3}$, &c., vanish at the extremity of life.

But this is not necessarily the case. For example, take De Moivre's law of mortality: and let the number (y) living at any age x, be $C(N-x)$; or $C(85-x)$. Then

$$y = C(N-x) \quad \therefore \frac{dy}{dx} = -C.$$

In this case then, neglecting interest, the mortality curve reduces to a straight line, cutting the axis in the point Z, at an angle whose tangent is C.

Also,

$$\mu = -\frac{1}{y}\frac{dy}{dx} = \frac{C}{C(N-x)} = \frac{1}{N-x}.$$

As x increases up to $N-1$, this increases to 1: then as x further increases from $N-1$ to N, μ increases from 1 to ∞. This at first sight appears an anomalous result, since the chance of dying in any assigned time can only be unity; but when we approach the extremity of life, we can take an age as close as we like to the extreme age which none of the lives actually attain, and the probability of dying in the following small interval is certainty. This being true, however small we take that interval, μ must increase without limit and ultimately become infinite; for, $\mu\delta x$ being the proportion which die in the small time δx, and the limit of this being unity when δx vanishes, the limit of μ must be ∞.

In the general case we have

$$\frac{dy}{dx} = \frac{d}{dx}(v^x l_x) = v^x l'_x - \delta v^x l_x,$$

where $\delta = \log_e(1+i) = -\log_e v$. At the extreme age in the table (z), $l_z = 0$, and $\left(\frac{dy}{dx}\right)_z = v^z l'_z$; which will not vanish unless $l'_z = 0$. It will be found that this is not generally the case; but its value will almost always be very small. Taking as before the Experience table, difference as follows the numbers living at successive ages.

x	l_x	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
100	0	1	2	4	5	−1	−1	−3	− 2
99	1	3	6	9	4	−2	2	−5	+11
98	4	9	15	13	2	0	−3	+6	− 9
97	13	24	28	15	2	−3	+3	−3	+ 3
96	37	52	43	17	−1	0	0	0	
95	89	95	60	16	−1	0	0		
94	184	155	76	15	−1	0			
93	339	231	91	14	−1				
92	570	322	105	13					
91	892	427	118						
90	1319	545							
89	1864								

Denote the leading differences here by d_1, d_2, d_3, &c., so that $d_1 = 1$, $d_2 = 2$, &c.; then if x be measured from the extreme age (100) we have by the ordinary formula of interpolation

$$l_{100-x}=l_{100}+xd_1+\frac{x(x-1)}{2}d_2+\frac{x(x-1)(x-2)}{1.2.3}d_3$$
$$+\frac{x(x-1)(x-2)(x-3)}{1.2.3.4}d_4+\dots.$$

We proceed as follows in order to determine the angle at which the curve representing the number living at any age cuts the axis of x, *i.e.* to find the value of $\left(\frac{d}{dx}l_{100-x}\right)_0$ or $-l'_{100}$.

We have

$$\frac{l_{100-x}-l_{100}}{x}=d_1+\frac{x-1}{2}d_2+\frac{(x-1)(x-2)}{2.3}d_3$$
$$+\frac{(x-1)(x-2)(x-3)}{2.3.4}d_4+\dots.$$

and making $x=0$, the limit of $\frac{l_{100-x}-l_{100}}{x}$, *i.e.*, $-l'_{100}$,

$$=d_1-\frac{d_2}{2}+\frac{d_3}{3}-\frac{d_4}{4}+\frac{d_5}{5}-\frac{d_6}{6}+\dots.$$

In the case of the Experience table, taking the first six differences as found above,

$$-l'_{100}=1-1+\frac{4}{3}-\frac{5}{4}-\frac{1}{5}+\frac{1}{6}$$
$$=\frac{1}{12}-\frac{1}{30}=\frac{1}{20}.$$

If we took two more differences, we should get a positive value of l'_{100}. This indicates that the curve of mortality cuts the axis of x at a point between the ages of 99 and 100; or that all the lives die before attaining an age which lies between 99 and 100. In consequence of l'_x not being generally zero at the extremity of life, the formula

$$a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

will not be accurately true. The amount of the error is however very small, except at very advanced ages. Its value will be considered more fully presently.

I now pass on to consider the nature of the curve which is substituted for PQ when we employ Mr. Woolhouse's formula; and I shall be able to show that the use of that formula is equivalent to substituting for each arc PQ of the true curve, the arc of a curve which touches PQ at the points P and Q, and of which the third differences are constant.

For this purpose, it will be convenient to review the steps of the demonstration by which that formula was arrived at. We start with the following formula, u_x being any function of x (see p. 207):—

$$\left.\begin{aligned}\frac{u_a}{2}+u_{a+1}+u_{a+2}+\ldots\ldots+u_{b-1}+\frac{u_b}{2}&=\int_a^b u_x dx\\ &+\frac{1}{12}\left\{\left(\frac{du_x}{dx}\right)_b-\left(\frac{du_x}{dx}\right)_a\right\}\\ &-\frac{1}{720}\left\{\left(\frac{d^3u_x}{dx^3}\right)_b-\left(\frac{d^3u_x}{dx^3}\right)_a\right\}\\ &+\ldots\ldots\ldots\end{aligned}\right\}\quad(4)$$

The formula will be more symmetrical if we put $l_x v^x$ for u_x instead of $l_{k+x}v^x$, as is done in the passage referred to. And $l_x v^x$ being the quantity tabulated for integral values of x under the heading D_x, it will be convenient to adopt the latter notation; also for brevity put D'_a for $\left(\frac{d}{dx}D_x\right)_a$, and D'''_a for $\left(\frac{d^3}{dx^3}D_x\right)_a$, &c.

Putting then $l_x v^x$, or D_x, for u_x in the above equation; and making a equal to k, and b equal to z,—the extreme age in the table of mortality which none of the lives actually attain—(*e.g.* in the Experience table, $z=100$)—the first member of the equation becomes $\frac{1}{2}D_k+D_{k+1}+D_{k+2}+\ldots+D_{z-1}$ or $D_k(\frac{1}{2}+a_k)$, and we get

$$D_k(\tfrac{1}{2}+a_k)=\int_k^z D_x dx+\frac{1}{12}(D'_z-D'_k)-\frac{1}{720}(D'''_z-D'''_k)+\ldots\ldots\quad(5)$$

Next put $l_{\frac{x}{m}}.v^{\frac{x}{m}}$ for u_x in the equation (4), and make a equal to mk, and b to mz, then the first member becomes $\frac{1}{2}D_k+D_{k+\frac{1}{m}}+D_{k+\frac{2}{m}}+\ldots\ldots+D_{z-\frac{1}{m}}$ or $D_k(\frac{1}{2}+ma_k^{(m)})$; and we get

$$\begin{aligned}D_k(\tfrac{1}{2}+ma_k^{(m)})&=\int_{mk}^{mz}D_{\frac{x}{m}}dx+\frac{1}{12}\left\{\left(\frac{d}{dx}D_{\frac{x}{m}}\right)_{mz}-\left(\frac{d}{dx}D_{\frac{x}{m}}\right)_{mk}\right\}\\ &\qquad-\frac{1}{720}\left\{\left(\frac{d^3}{dx^3}D_{\frac{x}{m}}\right)_{mz}-\left(\frac{d^3}{dx^3}D_{\frac{x}{m}}\right)_{mk}\right\}\\ &\qquad+\ldots\ldots\ldots\ldots\\ &=m\int_k^z D_x dx+\frac{1}{12m}(D'_z-D'_k)-\frac{1}{720m^3}(D'''_z-D'''_k)\\ &\qquad+\ldots\ldots\end{aligned}$$

and dividing by m,

$$D_k\left(\frac{1}{2m}+a_k^{(m)}\right)=\int_k^z D_x dx+\frac{D'_z-D'_k}{12m^2}-\frac{D'''_z-D'''_k}{720m^4}+\ldots.$$

Now subtracting (5) from this, we have

$$D_k\left\{a_k^{(m)}-a_k-\frac{m-1}{2m}\right\}=\frac{m^2-1}{12m^2}(D'_k-D'_z)$$
$$-\frac{m^4-1}{720m^4}(D'''_k-D'''_z)$$
$$+\quad\ldots.$$

and

$$a_k^{(m)}=a_k+\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\cdot\frac{D'_k-D'_z}{D_k}-\frac{m^4-1}{720m^4}\cdot\frac{D'''_k-D'''_z}{D_k}+\ldots.$$

It will be shown presently how the values of D'_k, D'''_k . . . may be found from the tabulated values of D_x.

Mr. Woolhouse's approximation is equivalent to

$$a_k^{(m)}=a_k+\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\frac{D'_k}{D_k}$$

For $D'_k=\left(\frac{d}{dx}l_xv^x\right)_k=l_k\log v\,.\,v^k+l'_kv^k$; and $\frac{D'_k}{D_k}=\log v+\frac{l'_k}{l_k}$ $=-(\delta+\mu_k)$. But a more correct approximation will be

$$a_k^{(m)}=a_k+\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\frac{D'_k-D'_z}{D_k}\quad.\quad.\quad.\quad.\quad(6)$$

since, as already stated, D'_z is not zero generally, although its value is very small. This formula would be strictly correct if $D'''_k-D'''_z=0$, $D^{\text{v}}_k-D^{\text{v}}_z=0$, &c.: *i.e.* if the third differences of D_x were constant from the age k to the end of the table. But this is not an admissible supposition, since it will not be generally possible so to fix the differences that over a long series of years the third differences may be constant. It will almost always be found that if the differences are so fixed as to make the substituted curve coincide closely with the original at certain points, it will at other points depart widely from it. But this may be done for a short distance without material error, as for example between any two consecutive ages x and $x+1$; and we will now consider the effect of that supposition.

Instead of commencing with the equation (4) we take the following more general one :—

$$\frac{u_a}{2}+u_{a+h}+u_{a+2h}+\ldots+u_{b-h}+\frac{u_b}{2}=\frac{1}{h}\int^b u_xdx+\frac{h}{12}(u'_b-u'_a)$$
$$+\frac{h^3}{720}(u'''_b-u'''_a)$$
$$+\quad\ldots.$$

Here let us make $a=k$, $b=k+1$, $h=\frac{1}{m}$; and $u_x=D_x$ as before; then

$$\tfrac{1}{2}D_k+D_{k+\frac{1}{m}}+D_{k+\frac{2}{m}}+\ldots D_{k+\frac{m-1}{m}}+\tfrac{1}{2}D_{k+1}=m\int_k^{k+1}D_x dx$$

$$+\frac{D'_{k+1}-D'_k}{12m}$$

$$-\frac{D'''_{k+1}-D'''_k}{720m^3}$$

$$+\ \ldots\ldots$$

Now Mr. Woolhouse's approximation is equivalent to neglecting the term $\frac{D'''_{k+1}-D'''_k}{720m^3}$ and all the following terms; that is, to substituting for D_x between the values k and $k+1$, a function $\mathbb{D}_x$, which has its third differences constant, and such that $\mathbb{D}_k=D_k$, $\mathbb{D}_{k+1}=D_{k+1}$, $\mathbb{D}'_k=D'_k$, $\mathbb{D}'_{k+1}=D'_{k+1}$, so that we have the following equation *accurately* true:—

$$\tfrac{1}{2}D_k+\mathbb{D}_{k+\frac{1}{m}}+\mathbb{D}_{k+\frac{2}{m}}+\ldots+\mathbb{D}_{k+\frac{m-1}{m}}+\tfrac{1}{2}D_{k+1}=m\int_k^{k+1}\mathbb{D}_x dx+\frac{D'_{k+1}-D'_k}{12m}.$$

But the values of $\mathbb{D}_x$ can differ very slightly from those of D_x between the limits supposed; so that we have very approximately

$$\tfrac{1}{2}D_k+D_{k+\frac{1}{m}}+D_{k+\frac{2}{m}}+\ldots+D_{k+\frac{m-1}{m}}+\tfrac{1}{2}D_{k+1}$$

$$=m\int_k^{k+1}D_x dx+\frac{D'_{k+1}-D'_k}{12m}.$$

Similarly

$$\tfrac{1}{2}D_{k+1}+D_{k+1+\frac{1}{m}}+\ldots+\tfrac{1}{2}D_{k+2}=m\int_{k+1}^{k+2}D_x dx+\frac{D'_{k+2}-D'_{k+1}}{12m},$$

and similar equations for each age, the last being

$$\tfrac{1}{2}D_{s-1}+D_{s-1+\frac{1}{m}}+\ldots+D_{s-\frac{1}{m}}=m\int_{s-1}^{s}D_x dx+\frac{D'_s-D'_{s-1}}{12m}.$$

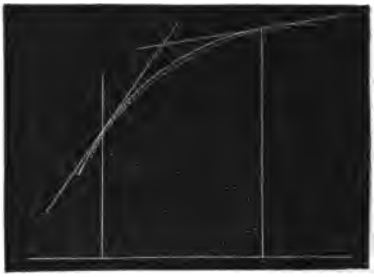
Adding all these equations together, we get

$$D_k(\tfrac{1}{2}+ma_k^{(m)})=m\int_k^s D_x dx+\frac{D'_s-D'_k}{12m},$$

whence we get, as before,

$$a_k^{(m)}=a_k+\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\,\frac{D'_k-D'_s}{D_k}.$$

In order still further to elucidate this point, I will now pursue the reverse process, and suppose a curve drawn from P to Q of the natûre above described, and investigate its properties independently.



Thus, let us assume, x being less than unity, that

$$u_{k+x}=u_k+Ax+Bx^2+Cx^3 \quad . \; . \; (7)$$

Then, differentiating,

$$u'_{k+x}=A+2Bx+3Cx^2.$$

Making $x=0$, $x=1$, successively, we get from the supposed conditions

$$u'_k=A, \quad u_{k+1}=u_k+A+B+C, \quad u'_{k+1}=A+2B+3C.$$

Hence

$$B+C=u_{k+1}-u_k-u'_k$$

$$2B+3C=u'_{k+1}-u'_k,$$

from which we find

$$B = 3u_{k+1}-3u_k-u'_{k+1}-2u'_k$$

$$C=-2u_{k+1}+2u_k+u'_{k+1}+u_k.$$

Then substituting for A, B, C, in (7) these values, and collecting the terms, we have

$$u_{k+x}=u_k(1-3x^2+2x^3)+u_{k+1}(3x^2-2x^3)$$
$$+u'_k(x-2x^2+x^3)+u'_{k+1}(-x^2+x^3) \quad . \; . \; . \; . \; (8)$$

Integrating, we get

$$\int u_{k+x}dx=u_k\left(x-x^3+\frac{x^4}{2}\right)+u_{x+1}\left(x^3-\frac{x^4}{2}\right)$$
$$+u'_k\left(\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right)+u'_{k+1}\left(-\frac{x^3}{3}+\frac{x^4}{4}\right)+C,$$

and

$$\int_0^1 u_{k+x}dx=\tfrac{1}{2}(u_k+u_{k+1})+\tfrac{1}{12}(u'_k-u'_{k+1}).$$

Again, making $x=\frac{1}{m}, \frac{2}{m}, \frac{3}{m} \ldots, \frac{m-1}{m}$, successively in the equation (7), we have

$$\tfrac{1}{2}u_k=\tfrac{1}{2}u_k$$

$$u_{k+\frac{1}{m}}=u_k+A\frac{1}{m}+B\frac{1}{m^2}+C\frac{1}{m^3}$$

$$u_{k+\frac{2}{m}}=u_k+\mathrm{A}\frac{2}{m}+\mathrm{B}\frac{4}{m^2}+\mathrm{C}\frac{8}{m^3}$$

$$\vdots \qquad\qquad \vdots$$

$$u_{k+\frac{m-1}{m}}=u_k+\mathrm{A}\cdot\frac{m-1}{m}+\mathrm{B}\frac{(m-1)^2}{m^2}+\mathrm{C}\frac{(m-1)^3}{m^3}$$

$$\tfrac{1}{2}u_{k+1}=\tfrac{1}{2}u_{k+1}.$$

Adding these equations together, we have

$$\tfrac{1}{2}u_k+u_{k+\frac{1}{m}}+u_{k+\frac{2}{m}}+\ .\ .\ .\ +u_{k+\frac{m-1}{m}}+\tfrac{1}{2}u_{k+1}$$

$$=\frac{u_k+u_{k+1}}{2}+(m-1)u_k+\frac{m-1}{2}\mathrm{A}+\frac{(m-1)(2m-1)}{6m}\mathrm{B}+\frac{(m-1)^2}{4m}\mathrm{C}$$

$$=\frac{u_k+u_{k+1}}{2}+(m-1)u_k+\frac{m-1}{2}u'_k$$

$$+\frac{(m-1)(2m-1)}{6m}(3u_{k+1}-3u_k-u'_{k+1}-2u'_k)$$

$$+\frac{(m-1)^2}{4m}(-2u_{k+1}+2u_k+u'_{k+1}+u'_k)$$

$$=\frac{m}{2}(u_k+u_{k+1})+\frac{m^2-1}{12m}(u'_k-u'_{k+1}).$$

Writing down the corresponding equations when $k+1$, $k+2$, . . . $z-1$, are respectively substituted for k,

$$\tfrac{1}{2}u_{k+1}+u_{k+1+\frac{1}{m}}+\ .\ .\ .\ +\tfrac{1}{2}u_{k+2}=\frac{u_{k+1}+u_{k+2}}{2}+\frac{m^2-1}{12m}(u'_{k+1}-u'_{k+2})$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\tfrac{1}{2}u_{z-1}+u_{z-1+\frac{1}{m}}+\ .\ .\ .\ +u_{z-\frac{1}{m}}=\frac{m}{2}\cdot u_{z-1}+\frac{m^2-1}{12m}(u'_{z-1}-u'_z),$$

and adding all these together,

$$u_k\{\tfrac{1}{2}+ma_k^{(m)}\}=mu_k\{\tfrac{1}{2}+a_k\}+\frac{m^2-1}{12m}(u'_k-u'_z),$$

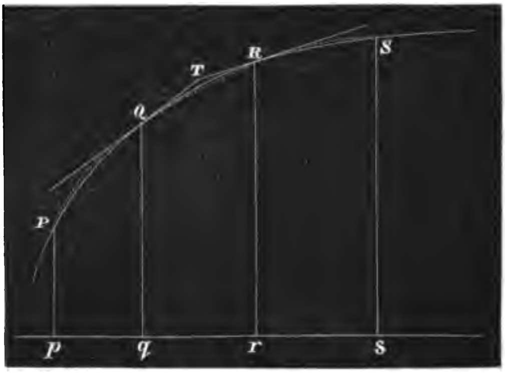
whence

$$a_k^{(m)}-a_k=\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\frac{u'_k-u'_z}{u_k},$$

which is the approximate formula (6) already found, u_k being here put for D_k.

The preceding investigation suggests an improvement upon the ordinary method of interpolation when constant third differences are used.

Let P, Q, R, S, be four points in any curve, the ordinates of which are equidistant, so that $pq=qr=rs$; and let it be required



to interpolate betwen Qq and Rr any number of ordinates proceeding by constant third differences; then the ordinary method of interpolation (which was pursued by myself on p. 202) is equivalent to substituting for QR a curve line of the third order, which, when continued in both directions, passes through P on the one side and S on the other. Instead of this, the substituted curve, according to the preceding process, has the same tangents as the original curve at the points Q and R; and may be expected to coincide more closely with it.

If constant fifth differences are used, the coincidence can be made still more complete by substituting for QR a curve of the fifth order whose equation is

$$u_{k+x}=u_k+Ax+Bx^2+Cx^3+Dx^4+Ex^5,$$

where A, B, C, D, E, are determined by the conditions that u_{k+1}, u'_k, u'_{k+1}, u''_k, u''_{k+1}, are to be the same in the substituted as in the original curve.

Thus we have

$$u'_{k+x}=A+2Bx+3Cx^2+4Dx^3+5Ex^4$$
$$u''_{k+x}=2B+6Cx+12Dx^2+20Ex^3,$$

and making $x=0$ and $x=1$ successively,

$$u_{k+1}=A+B+C+D+E$$
$$u'_k\ =A$$
$$u'_{k+1}=A+2B+3C+4D+5E$$
$$u''_k\ =2B$$
$$u''_{k+1}=2B+6C+12D+20E$$

—five equations, from which the five unknowns may be found without difficulty.

It is here supposed that the values of u_x are given for a series of equidistant points, corresponding to the values of x, $k-rt$, $k-(r-1)t$, ... $k-2t$, $k-t$, k, $k+t$, $k+2t$ .... and that the values of u'_k, u'_{k+1}, u''_k, u''_{k+1}, are found from these values of u_x by means of the formula given by Mr. Woolhouse, vol. xi., p. 68, to which I have already several times referred.

I now proceed to examine as to the magnitude of the neglected term $-\frac{m^2-1}{12m^2}\cdot\frac{D'_z}{D_k}$.

We have $D_x=l_xv^x$,

$$\therefore \qquad D'_x=l'_xv^x-\delta l_xv^x.$$

Putting z for x, since $l_z=0$,

$$D'_z=l'_zv^z,$$

and

$$\frac{D'_z}{D_k}=\frac{l'_zv^z}{l_kv^k}=v^{z-k}\cdot\frac{l'_z}{l_k}.$$

Now we have seen that in the Experience table l'_z, *i.e.* l'_{100}, $=-\frac{1}{20}$. If we put $k=90$, and take interest at 3 per cent.,

$$\frac{D'_z}{D_k}=v^{z-k}\cdot\frac{l'_z}{l_k}=-\frac{v^{10}}{l_{90}}\times\frac{1}{20}$$

$$=-\frac{\cdot74409}{1319}\times\frac{1}{20}$$

and

$$-\tfrac{1}{12}\frac{D'_{100}}{D_{90}}=+\cdot00000235.$$

Similarly, putting $k=95$, we get

$$-\tfrac{1}{12}\frac{D'_{100}}{D_{95}}=\frac{\cdot8626}{12\times89\times20}=\cdot000040,$$

and we see that the value of the term is insignificant except for ages over 90; and we may in practice safely rely on the values as given to five decimal places by the formula

$$a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta).$$

I will conclude with an example of the method of finding the values of D'_k, D''_k, &c., from the tabulated values of D_k, D_{k+1}, &c.

Taking the Experience 3 per cent. table, as given by Jenkin

Jones, making $k=85$, and differencing the values of D_x, we get, using the notation of Mr. Woolhouse, as followed in p. 210,

$$a_0 = -109{\cdot}63972$$
$$c_0 = -\ \ 1{\cdot}63248$$
$$e_0 = \ \ {\cdot}19548$$
$$g_0 = \ \ {\cdot}32359$$

$$D'_{85} = a_0 - \frac{c_0}{6} + \frac{e_0}{30} - \frac{g_0}{140} = -109{\cdot}35881$$

$$D'''_{85} = c_0 - \frac{e_0}{4} + \frac{7g_0}{120} \quad = \quad - \quad 1{\cdot}70023$$

whence $\quad \dfrac{D'_{85}}{D_{85}} = -{\cdot}24903$

and $\quad \dfrac{D'''_{85}}{D_{85}} = -{\cdot}003871$

Now $D_x = v^x l_x$,

$$\therefore\ D'_x = \log_e v\,.\,v^x l_x + v^x l'_x$$

and $\quad \dfrac{D'_x}{D_x} = \log_e v + \dfrac{l'_x}{l_x} = -\delta - \mu_x.$

Similarly it may be shown that

$$\frac{D''_x}{D_x} = -\delta^3 - 3\delta^2\mu_x - \frac{3\delta l''_x}{l_x} + \frac{l'''_x}{l_x}.$$

Thus then, referring to p. 211, it will be seen that we there found the value of $-\dfrac{D'_{85}}{D_{85}}$ to be ·24905 and that of $-\dfrac{D'''_{85}}{D_{85}}$ to be ·003820. It will be noticed that these values coincide closely, though not exactly, with the values there found by a widely different process. An exact coincidence is not to be expected.

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On the Law of Mortality. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 29th April, 1867.]

IN the following pages I shall have frequent occasion to avail myself of a term which the progress of the analysis of life contingencies has rendered indispensable, but which is not found in any of the standard elementary works in that science. I think, therefore, that I cannot better commence this paper than by an attempt to give an explanation of the expression "force of mortality," sufficiently ample to obviate any difficulties which might otherwise be experienced on this score.

In the subjoined table L_x denotes the number living at age x in a mortality table, and ΔL_x the difference corresponding to an increment of Δx in the age—in this case 10 years. The other characters will be explained further on.

x. (1.)	L_x. (2.)	$-\Delta L_x$. (3.)	$-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}$. (4.)	$\frac{L_x - L_{x+1}}{L_x}$. (5.)	$-\frac{1}{L_x}\cdot\frac{dL_x}{dx}$. (6.)	x. (7.)
20	9626·100	754·934	·00784	·00775	·00775	20
30	8871·166	818·836	·00923	·00875	·00872	30
40	8052·330	998·874	·01241	·01108	·01097	40
50	7053·456	1370·050	·01942	·01653	·01626	50
60	5683·406	1918·892	·03376	·02919	·02868	60
70	3764·514	2190·060	·05818	·05826	·05780	70
80	1574·454	1350·552	·08578	·12310	·12614	80
90	223·902	221·368	·09887	·25824	·28647	90
100	2·534	2·534	·10000	·49913	·66265	100

The second column then, headed L_x, contains the numbers living at successive decennial intervals of age; the third $(-\Delta L_x)$ shows the decennial decrements; the fourth $\left(-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}\right)$ gives the ratio which the average annual decrement of each decennial period bears to the number living at the commencement of the period; while the fifth column $\left(\frac{L_x-L_{x+1}}{L_x}\right)$ exhibits the series known as the rate of mortality—*i.e.*, the ratio of the number actually dying in one year to the number living at the commencement of that year.

Thus, out of 9626·100 persons living at age 20, 754·934 die before attaining the age of 30—which is at the *average* rate of 75·4934 per annum. Dividing this last by 9626·100, we have ·00784 for the ratio which the average number of annual deaths bears to the number living at the commencement of the decennial interval.

Comparing this with the corresponding number in column 5, we see that it is but little in excess of the rate of mortality for the least of the 10 ages included in the given interval—the rate of mortality at age 20 being ·00775. The same comparative deficiency in the average annual proportion of deaths of the decennial interval will be observed upon comparing the two columns for the respective ages of 30, 40, 50, and 60. At the age of 70 and upwards, however, the deficiency in question becomes much more apparent—the average annual rate of decrease, during the 10 years, being then actually *less* than the rate of mortality for the youngest age of the decennial period; and so rapidly does this go on, that at the age of 100 the former is little more than one-fifth of the latter.

It is scarcely necessary to point out that this result is owing to the fact that as the numbers living are gradually diminishing by the effect of mortality, the number of deaths must also experience a corresponding diminution; and the longer the interval between the ages observed, the greater must be the disturbance caused by this incessant reduction in the number exposed to the risk of death. My object, however, in calling attention to a phenomenon which must be sufficiently familiar to all who have given the least consideration to the subject, is to show how extremely inadequate the average annual number of deaths—taken for several years together—becomes as a measure of the actual *intensity* or *force* of the operating causes by which the decrements of life are produced.

Now, it will be observed that the function $\frac{L_x - L_{x+1}}{L_x}$ (representing the rate of mortality), of which the values are shown in col. 5, is of precisely the same form as the function $-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta_x}$ $\left(\text{or } -\frac{1}{L_x}\cdot\frac{\Delta L_x}{10}\right)$, from which the values in col. 4 are computed. For the latter is equivalent to $\frac{L_x - L_{x+10}}{10 L_x}$, which becomes identical with the former by substituting 1 for 10 as the increment of x. But it is evident that the same causes which render the series in col. 4 so imperfect a measure of the force of mortality must also tend (in a less degree) to make the series in col. 5 unfit for the same purpose. If, therefore, instead of making Δx (or the increment of age) $=1$ in the formula $-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}$, we make it a fraction (say ·5, for instance), we shall obtain a still better expression for the measure in question. And if we go still further, and diminish Δx without limit, we shall evidently get rid of the disturbing element altogether, and thus obtain a perfect measure of the actual intensity or force of mortality at each age.

This is, in effect, the process adopted in the construction of col. 6; $\frac{dL_x}{dx}$ (which in the language of the differential calculus is called the differential coefficient of L_x) being the limit of the ratio of the infinitely small decrement of L_x to the infinitely small increment of x, by which it is produced. The means by which the limit in question may be computed will sufficiently appear in the course of the following pages.

The considerations to which I have thus ventured to call attention afford, I think, a sufficient explanation of the nature of the function known as the "force of mortality," as well as of the reason for its designation; and at the same time convey some idea of the importance which such a function is calculated to possess in the investigation of the nature of the law of mortality, and the analysis of life contingencies generally.

I proceed now to establish some theorems in connection with the expression $-\frac{1}{L_x}\cdot\frac{dL_x}{dx}$, which will be of service to us in the sequel; premising that when finite differences are used they are (unless the contrary be expressed) to be taken from the series L_x, L_{x+n}, L_{x+2n}, &c.—in other words, n is substituted for Δx.

I. The usual expansion of $\frac{dL_x}{dx}$ in terms of the finite differences is (De Morgan's *Diff. and Int. Calculus.*, p. 264)

$$n\cdot\frac{dL_x}{dx}=\Delta L_x-\tfrac{1}{2}\Delta^2L_x+\tfrac{1}{3}\Delta^3L_x-\ldots.$$

Transforming the second member of this equation into the system of central differences (see Mr. Woolhouse's paper on Interpolation, *Assurance Magazine*, vol. xi.), we have

$$n\frac{dL_x}{dx}=\frac{\Delta L_x+\Delta L_{x-n}}{2}-\frac{1}{2.3}\,\frac{\Delta^3L_{x-n}+\Delta^3L_{x-2n}}{2}$$
$$+\frac{1.2}{3.4.5}\cdot\frac{\Delta^5L_{x-2n}+\Delta^5L_{x-3n}}{2}-\ldots.$$

Hence, neglecting third and higher differences, we get for an approximate expression for the force of mortality $-\frac{\Delta L_x+\Delta L_{x-n}}{2nL_x}$. If n be taken $=1$, we have the formulæ given by Mr. Woolhouse in his excellent paper above referred to.

The following still more rapidly converging series may be used with advantage when the values of $L_{x+n:2}$, $L_{x+3n:2}$, &c. ($n:2$ denoting the quantity $\frac{n}{2}$, &c.) are contained in the table:—

$$n\frac{dL_x}{dx}=\Delta L_{x-n:2}-\frac{1}{24}\Delta^3L_{x-3n:2}+\frac{3}{640}\Delta^5L_{x-5n:2}-\ldots.\text{*}$$

II. If the force of mortality be denoted by F_x, we shall have $-dL_x=L_xF_xdx$. Now it will be shown further on, that whether a population be stationary or fluctuating—and by whatever law, in the latter case, the fluctuations may be governed—the ratio of the the annual average number of deaths, during a period of n years, between the ages v and $v+n$, to the mean or average population during the same period and between the same ages, is accurately represented by the formula

$$\frac{-\int_0^n\frac{dL_{v+x}}{dx}C_xdx}{\int_0^nL_{v+x}.C_x.dx},$$

where C_x is a function of x depending upon the nature of the fluctuations of the population during the period and between the ages observed, and which becomes a constant and disappears when the population is stationary. But from the equation $-dL_x = L_xF_xdx$ we have $-dL_{v+x}C_x = L_{v+x}F_{v+x}C_xdx$, and therefore $-\int_0^n \frac{dL_{v+x}}{dx}C_xdx = \int_0^n L_{v+x}F_{v+x}C_xdx$. Now if F_{v+x} be supposed constant between the limits of integration, and equal to its value at the mean age $v+\frac{1}{2}n$, the second member of the last equation becomes $F_{v+\frac{1}{2}n}\int_0^n L_{v+x}C_xdx$, and we have

---

* The higher differential coefficients may be similarly developed. For instance, we have $\frac{n^2}{2}\cdot\frac{d^2L_x}{dx^2}=\frac{\Delta^2L_{x-n}}{2}-\frac{1}{2.3}\frac{\Delta^4L_{x-2n}}{4}+\frac{1.2}{3.4.5}\cdot\frac{\Delta^6L_{x-3n}}{6}-\ldots.$ Of course L_x may here stand, generally, for a function of x.

$$F_{v+\frac{1}{2}n} = \frac{-\int_0^n \frac{dL_{v+x}}{dx}C_xdx}{\int_0^n L_{v+x}C_xdx},$$

so that the ratio of the average annual number of deaths to the mean population between given ages approximately represents the force of mortality at the mean age.

III. Our opening remarks will have made it sufficiently apparent that the force of mortality expresses the rapidity with which, at any given instant of time, a body of individuals of a given age are diminishing by death. Being a function of the age it is incessantly varying in passing from one age to the next, but it may be accurately defined as the ratio between the number living at the given age and the number of deaths which would take place in one year, supposing the force of mortality to remain constant during that period, and each vacancy arising by death to be filled up as it occurs by the substitution of a life of the same age. When speaking of the general mortality, irrespective of the particular causes, it may be designated as the "total" force of mortality, to distinguish it from the "partial" forces, by which latter term I propose to designate the ratio above defined, when we are considering the deaths arising from one or more particular diseases only. The total force of mortality at age x I have denoted by F_x, and similarly the several partial forces will be represented by F'_x, F''_x, &c.

IV. Let the total force of mortality in a given body of individuals be composed of the two distinct and independent partial forces F'_x and F''_x. And as L_x denotes the number living at age x in a body subject to the total force of mortality, let L'_x and L''_x respectively represent the number living in a body subject to each one of the partial forces only. And let it be supposed that the decrement $-\Delta L'_x$ (corresponding to any small increment of time Δ_x) takes place at the commencement of the interval, and the decrement $-\Delta L''_x$ at the termination thereof. Then the total decrement of L_x (supposing both causes of mortality to be in operation together) will be

$$-\Delta L_x = -L_x \times \frac{\Delta L'_x}{L'_x} - L_x\left(1-\frac{\Delta L'_x}{L'_x}\right)\frac{\Delta L''_x}{L''_x}$$

$$= -L_x\left(\frac{\Delta L'_x}{L'_x}+\frac{\Delta L''_x}{L''_x}\right) + L_x . \frac{\Delta L'_x}{L'_x} . \frac{\Delta L''_x}{L''_x} .$$

Now, by taking Δx sufficiently small, the supposition as to the time at which the decrements take place may be made as near the truth as we please. Let Δx be diminished without limit, in which case $\Delta L'_x . \Delta L''_x$ becomes infinitely small in comparison with $\Delta L'_x$ and $\Delta L''_x$, and the term in which it is a factor may therefore be neglected without affecting the truth of the equation. Dividing the remaining terms by $L_x dx$, we have

$$-\frac{dL_x}{L_x dx} = -\frac{dL'_x}{L'_x dx} - \frac{dL''_x}{L''_x dx}$$

or $F_x = F'_x + F''_x$.

Hence, generally the total force of mortality is equal to the sum of the several partial forces.

V. Integrating both sides of the equation $\frac{dL_x}{L_x dx} = \frac{dL'_x}{L'_x dx} + \frac{dL''_x}{L''_x dx}$, we have $\log L_x = \log L'_x + \log L''_x + c$, whence $gL_x = L'_x L''_x$. But a table of the values of gL_x, and one of L_x, are, for all practical purposes, the same. From this it appears that if we have tables of L'_x, L''_x, L'''_x, &c., showing the numbers living at successive ages in several bodies of individuals, each of which is supposed to be subject only to one given cause of death, we may construct a table of the numbers living in a body subject to all or to any given number of such causes by merely multiplying the number living at each age, in the separate tables, into each other. Henceforth, then, L_x will denote $L'_x . L''_x \ldots$

VI. Representing by $-\Delta' L_x$ the number dying from one particular cause (viz., that corresponding to L'_x) out of L_x persons subject simultaneously to several causes of mortality, let it be required to find an expression for its value. The probability of a life aged v dying in the instant immediately following the expiration of the time x is found by multiplying the probability of the life surviving the period x into the probability of a life aged $v+x$ dying instantaneously. We have, therefore, for the probability required, $-\frac{L_{v+x}}{L_v} . \frac{dL'_{v+x}}{L'_{v+x}}$, or $-\frac{1}{L_v} . L''_{v+x} . dL'_{v+x}$. Integrating from $x=0$

to $x=n$, we have for the probability of the life dying within n years, from the given cause, $-\frac{1}{\mathrm{L}_v}.\int_0^n \mathrm{L}''_{v+x}.\frac{d\mathrm{L}'_{v+x}}{dx}dx$, or $-\frac{1}{\mathrm{L}_v}\int_v^{v+n}\mathrm{L}''_x\frac{d\mathrm{L}'_x}{dx}dx$, and multiplying by L_v, we find that

$$-\Delta'\mathrm{L}_v=-\int_v^{v+n}\mathrm{L}''_x.\frac{d\mathrm{L}'_x}{dx}dx.$$

Hence also
$$-\Delta''\mathrm{L}_v=-\int_v^{v+n}\mathrm{L}'_x.\frac{d\mathrm{L}''_x}{dx}dx.$$

Adding together the corresponding sides of these equations, the sums should be equal. Now $\int\mathrm{L}''_x.d\mathrm{L}'_x+\int\mathrm{L}'_x.d\mathrm{L}''_x=\mathrm{L}'_x\mathrm{L}''_x=\mathrm{L}_x$, and taking the limits $x=v$ to $x=v+n$, we get $\mathrm{L}_{v+n}-\mathrm{L}_v$ or $\Delta\mathrm{L}_v$. Therefore

$$-\Delta'\mathrm{L}_v-\Delta''\mathrm{L}_v=-\Delta\mathrm{L}_v,$$

the two members of which equation are evidently identical.

The value of $-\Delta'\mathrm{L}_x$ may be deduced in a somewhat different form, for $-\frac{\mathrm{L}_{v+x}}{\mathrm{L}_v}\frac{d\mathrm{L}'_{v+x}}{\mathrm{L}'_{v+x}}=\frac{\mathrm{L}_{v+x}\mathrm{F}'_{v+x}}{\mathrm{L}_v}dx$. Integrating and multiplying by L_v gives us

$$-\Delta'\mathrm{L}_v=\int_v^{v+n}\mathrm{L}_x\mathrm{F}'_x dx.$$

VII. The probability of dying (from the given cause) in the infinitely small interval of time dx being $-\frac{d\mathrm{L}'_x}{\mathrm{L}'_x}$, the infinitely small decrement $(-d'\mathrm{L}_x)$ from the deaths resulting from this particular force of mortality, in the number L_x, is $-\mathrm{L}_x\cdot\frac{d\mathrm{L}'_x}{\mathrm{L}'_x}$ or $\mathrm{L}_x\mathrm{F}'_x dx$. And from the equation $-d'\mathrm{L}_x=\mathrm{F}'_x\mathrm{L}_x dx$ we get

$$\mathrm{F}'_x=-\frac{d'\mathrm{L}_x}{\mathrm{L}_x dx},$$

and also $-\Delta'\mathrm{L}_v=\int_v^{v+n}\mathrm{L}_x\mathrm{F}'_x dx$, the result already arrived at by a different process in (VI.).

VIII. From the equation $-d'\mathrm{L}_x=\mathrm{L}_x\mathrm{F}'_x dx$ we get the approximate equation $\mathrm{F}'_{v+\frac{1}{2}n}=\frac{\int_0^n\frac{d'\mathrm{L}_{v+x}}{dx}\cdot\mathrm{C}_x dx}{\int_0^n\mathrm{L}_{v+x}\mathrm{C}_x dx}$ by the same process as that adopted in II. We see then that the theorem established in that article, viz., "That the average annual deaths in a given population (whether stationary or fluctuating) between the ages x and $x+n$, divided by the average number living between those ages, approximately represents the force of mortality at the mean age $x+\frac{1}{2}n$," is applicable as well to "partial" forces as to the "total" force of mortality.

In establishing the preceding theory of partial forces, it has been tacitly assumed that the numbers of the population are affected only by the births on the one hand and by the deaths on the other. But in all practical applications of the subject another important element, viz. migration, presents itself, so that it becomes necessary to show that the theorems hold good when that element is taken into consideration.

Let P_0 denote the number of annual births—which we will first suppose to be constant—and of these let there be P_n who will survive and remain in the country until the completion of n years of age. To simplify the case, let us assume, in the first instance, that there are no immigrants. Then if the law of mortality and migration remain constant (varying only with the age) during the period required for a whole generation to become extinct, the population will then have reached the stationary state. The number thenceforth annually completing the age x will be P_x, and $-\Delta P_x$ or P_x-P_{x+1} will represent the number annually disappearing by death and emigration between the ages x and $x+1$.

If, again, we take the case of a population increasing in geometrical progression—by reason of the annual births exceeding the annual disappearances in a constant ratio to the existing population, the numbers annually completing each successive year of age will be in the following proportions, viz., P_x, $P_{x+1}r^{-1}$, $P_{x+2}r^{-2}$, . . . where P_x, P_{x+1}, P_{x+2} . . . are the corresponding numbers in the case of a stationary population.

And as in the preceding theorems relative to the decrements of life no supposition was made as to the nature of the law of mortality, it is evident that they must all hold good in the case of the decrements resulting from death and emigration together—which for the sake of distinction we may term the decrements of population. And further, as we have seen that by means of the theory of partial forces of mortality we may investigate the law which regulates the deaths from one or more particular diseases, notwithstanding that the population may also be affected by deaths arising from other diseases; so we may with equal facility perform such investigations notwithstanding that the population may be affected by the disappearance of individuals by emigration, and this even although we may have no record of the number annually emigrating at each age.

To complete the subject we have now to include the case in which *im*migration as well as emigration takes place. If we assume that at every age the emigrants exceed the *im*migrants, the case has just been disposed of, for the whole of the immigrants may be supposed to take the place of an equal number of emigrants; and the *effective* force of emigration at each age is represented by the excess of the latter over the former. But it is evident, nevertheless, that emigration and immigration may also be considered as distinct forces, the one being a *positive* and the other a *negative* force; and this consideration leads us naturally to the case in which the immigrants may exceed the emigrants, which we may see is met by extending our conceptions of the force of migration as one which may be either positive or negative.

This mode of treating the subject gets rid of the necessity of investigating separately the effect of migration, which Mr. Milne has done, in considerable detail, in his able and elaborate work on life contingencies; and also renders unnecessary the limitation which the eminent writer referred to has introduced in some of his theorems, viz., that the population subject to investigation shall not be affected by migration. (See Milne, articles 184 and 188.)

We are now in a position to proceed with the investigation of the subject which forms the title of the present paper.

The formula which I have already had occasion to bring under the notice of this Institute, as representing very closely the normal law of human mortality, is derived from a modification of Mr. Gompertz's simple and highly ingenious theory that the power to oppose destruction loses equal proportions in equal times. The modification which I propose to introduce consists in the limitation of the theory to a *portion only* of the partial forces of mortality; and the assumption that the remaining partial forces operate (in the aggregate) with equal intensity at all ages. This modification will, I think, lose none of its claims to an impartial consideration, from the fact that it in no way interferes with the philosophical principle upon which Mr. Gompertz has shown his theory to be based: a feature which distinguishes his formula from all others which have hitherto been proposed, and which doubtless accounts for the favourable reception it has met with from the highest scientific authorities; from those in fact best qualified, from their habits of thought, to distinguish the "general" from the "particular"—the permanent and essential law from the temporary and accidental circumstances by which that law is obscured and modified.

From one or two quarters, indeed, Mr. Gompertz's hypothesis has met with an opposition which I can account for only by supposing the existence of a vague suspicion that the hypothesis in question is nothing more than a mere abstract theory, not derived from experiment, but to support which, on the contrary, it is necessary to do violence to the facts supplied from actual observation. How much opposed such a suspicion is to the truth, the following extracts from Mr. Gompertz's writings will testify:—"This equation between the number living and the age," he says in his first paper, "is deserving of attention, because it appears corroborated during a long portion of life by experience; as I derive the same equation from various published tables of mortality, during a long period of man's life, which experience therefore proves that the hypothesis approximates to the law of mortality during the same portion of life; *and in fact the hypothesis itself was derived from an analysis of the experience here alluded to.*" And again, speaking of the modification introduced by himself, for the purpose of extending his formula to the whole duration of life, he says, after giving the result of his experiments, "And I consider the equation quoted sufficiently interesting for my endeavour to discover the cause of its existence;" and further on he adds, "And contemplating on this law of mortality, I endeavoured to inquire if there could be any physical cause for its existence."

These extracts prove conclusively that this eminent man followed no such erroneous method as that supposed; but that, on the contrary, he confined himself strictly to the *interpretation* of the facts supplied by experiment. His method is in perfect accordance with the precepts of our great English philosopher, Bacon, of whose highly figurative and graphic style we are reminded by Mr. Gompertz's writings—a feature which renders their perusal quite refreshing in this oppressively logical age.

In the supplement to the 25th Annual Report of the Registrar-General are given—(1) the mean population of England (in decennial intervals of age) during the years 1851 to 1860, (2) the average annual number of deaths from different causes during the same period, at the corresponding ages, and (3) the ratios of the latter to the former, which we have shown to be the approximate value of the several partial forces which together make up the aggregate or total force of mortality. We have here, then, the means of testing the value of the proposed modification of Mr. Gompertz's theory (so far as the necessarily uncertain nature of the data will admit) by comparing the results with those deduced from the theory as originally propounded.

The formula for F_x, according to Mr. Gompertz's theory, is Bq^x. For this I propose to substitute $A+Bq^x$, where A is the sum of certain partial forces which we assume to be, in the aggregate, of equal amount at all ages. The quantity Bq^x may also consist of the aggregate of several forces of a similar nature. So that we may put

$$F_x=(a+a'+a''+\ldots)+(b+b'+b''+\ldots)q^x,$$

where $a+a'+a''+\ldots=A$, and $b+b'+b''+\ldots=B$.

I do not profess to be able to separate the whole category of diseases into the two classes specified—viz., diseases depending for their intensity solely upon the gradual diminution of the vital power, and those which depend upon other causes, the nature of which we do not at present understand. I apprehend that medical science is not sufficiently advanced to render such a desideratum possible of attainment at present. I propose only at present to show that there are certain diseases—and those too of a well-defined and strictly homogeneous character—which follow Mr. Gompertz's law far more closely than the aggregate mortality from all diseases taken together. I shall then have given sufficient reason for the substitution of the form $Bq^x+\phi(x)$ for the force of mortality in lieu of Bq^x: the proof that the terms included in $\phi(x)$ form, *in the aggregate,* a constant quantity, I shall leave until we come to the examination of data more satisfactory than the returns of population and the public registers of deaths.

The two following tables are taken from the supplement before referred to. They give, first, the number of annual deaths (from all causes) to 1,000,000 living; and secondly, the number of annual deaths from certain specified causes to the same number living. The causes of death, as well as the ages for which they are given, have of course been selected as the most favourable exponents of the law of geometrical progression; but it will be observed that the former embrace all the principal vital organs of the body, and the latter include the whole of the period from early manhood to the confines of extreme old age.

The column headed "total force of mortality" should form a geometrical progression if Gompertz's law were applicable thereto. That it does not, however, form such a progression, is evident by inspection; the rate of increase in the earlier terms being less than

50 per cent., and gradually increasing until it exceeds 100 per cent. A similar result is found in all the known tables when the law is applied to the *total* force of mortality, the remedy for which (in constructing mortality tables by Mr. Gompertz's formula) is usually sought in a change of the constants of the formula after certain intervals. It is this gradual but constant variation of the rate of increase *in one direction*, and the fact of its being uniformly found in *all* tables, that show unmistakeably that if the law itself be true, its application stands imperatively in need of some modification.

Male Life, 1851–60.

Ages.	Total Force of Mortality.	PARTIAL FORCES OF MORTALITY.					
		Lungs.	Heart.	Kidneys.	Stomach and Liver.	Brain.	Sum of five preceding Columns.
25–34	9,574	772	514	174	464	638	2,562
35–44	12,481	1,524	1,002	292	890	1,180	4,888
45–54	17,956	3,092	1,898	471	1,664	1,990	9,115
55–64	30,855	6,616	4,130	937	3,032	4,097	18,812
65–74	65,332	13,416	8,714	2,453	4,837	9,831	39,251

Female Life, 1851–60.

Ages.	Total Force of Mortality.	PARTIAL FORCES OF MORTALITY.					
		Lungs.	Heart.	Kidneys.	Stomach and Liver.	Brain.	Sum of five preceding Columns.
25–34	9,925	582	603	109	570	532	2,395
35–44	12,147	1,049	1,118	151	937	872	4,127
45–54	15,198	2,062	2,064	212	1,608	1,681	7,627
55–64	27,007	5,027	4,558	317	2,967	3,818	16,687
65–74	58,656	11,016	8,916	485	4,692	8,905	34,014

The modification which I have suggested, viz., there are certain partial forces of mortality (how many I do not pretend to say) which increase in intensity with the age in a constant geometrical ratio, while there are also certain other partial forces which do not so increase, may be tested by an examination of the six columns which follow that of the *total* force above referred to. The tendency to a geometrical progression is more or less apparent in all of them; the average rate of increase being such that the force of mortality somewhat more than doubles itself in 10 years.

It should be observed that, in addition to the diseases of the particular organs specified, other diseases of a kindred nature are also included under each of the above five partial forces. Possibly if more detailed information were accessible, we might be able to trace the geometrical character during a still more extended period of life. This, at least, I find to be the case in reference to one particular disease, viz., bronchitis, which in the preceding tables is included in the class of "lung diseases." Now it so happens that the deaths from bronchitis alone, for a long series of years, are given in the 26th Annual Report of the Registrar-General, from which the materials for the following table are taken. The number living is supposed to be 100,000, instead of 1,000,000 as in the two preceding tables.

Ages.	1848 to 1854. (7 Years.)		1855 to 1857. (3 Years.)		1858 to 1863. (6 Years.)	
	Males.	Females.	Males.	Females.	Males.	Females.
15–25	8	9	9	9	9	9
25–35	17	16	21	22	22	21
35–45	42	34	55	45	59	50
45–55	107	85	133	112	151	126
55–65	259	218	333	316	379	351
65–75	589	525	801	697	876	834
75–85	1,027	906	1,463	1,325	1,614	1,479

In the preceding examination of the results of the Registrar-General's returns of deaths, I have confined myself to the object of proving that Gompertz's law is traced much more distinctly in the deaths arising from certain specified diseases, than in the deaths arising from all causes together. If I have succeeded in this object (and I think it can scarcely be denied that I *have* succeeded), I have justified the introduction of an additional term in the formula representing the total force of mortality; but I have as yet brought forward nothing to show that such additional term is a constant in respect of the age, and varying only with the peculiar characteristics which distinguish different sets of observations from each other.

The several observations, however, which I now proceed to examine, if they do not enable us (like the former) to test particular terms of the function referred to, yet they will nevertheless afford a very satisfactory criterion of the complete expression. Not only, therefore, do they form by themselves (on account of their unquestionable accuracy and trustworthiness) ample evidence of the truth of the supposed law of mortality, but they also supply the deficiency, above adverted to, in the preceding investigation, as regards the requisite proof of the constancy of the term representing the aggregate of the remaining partial forces of mortality.

Commencing with the very valuable observations on the "Peerage Families" (both sexes), I find, by dividing the entire period of life into intervals of 14 years—neglecting, however, the first—the following results:—

$$\begin{array}{lcc}
\log L_{14} = \cdot 99034 & & \\
 & -\cdot 05068 & \\
\log L_{28} = \cdot 93966 & & -\cdot 00716 \\
 & -\cdot 05784 & \\
\log L_{42} = \cdot 88182 & & -\cdot 02559 \\
 & -\cdot 08343 & \\
\log L_{56} = \cdot 79839 & & -\cdot 11395 \\
 & -\cdot 19738 & \\
\log L_{70} = \cdot 60101 & & -\cdot 41273 \\
 & -\cdot 61011 & \\
\log L_{84} = \bar{1}\cdot 99090 & &
\end{array}$$

The tendency to a geometrical progression in the four terms of the second order of differences is sufficiently apparent. In order, however, to show this more distinctly, I have devised the following method of correcting the series $\log L_x$ so that the four terms in question shall form a perfect geometrical progression.

If the series consist of *five* terms, and consequently the second order of differences of *three*, the latter may be converted into a pure geometrical progression by substituting for the original series another of the following form, viz.,

$$\log L_0 + p, \ \log L_n - p, \ \log L_{2n} + p, \ \log L_{3n} - p, \ \log L_{4n} + p,$$

where p is derived from the equation

$$4p = \frac{(\Delta^2_n)^2 - \Delta^2_0 \times \Delta^2_{2n}}{\Delta^2_0 + 2\Delta^2_n + \Delta^2_{2n}}.\text{*}$$

This method, it is true, changes the value of the radix of the table, but I see no necessity for making a distinction between that and other terms of the series; for in comparing the terms of the *altered* with those of the *original* series, the object is to ascertain their bearing with respect to the original series *generally*, and not to any one term in particular. Secondly, by the method adopted, the first differences (which are the logarithms of the probabilities of living n years) are increased or diminished by an uniform quantity; whereas by omitting the correction in $\log L_0$, the first term of the first order of differences would be increased or diminished by one-half of the quantity introduced into the remaining terms. Lastly, the equation for p would be of the second order, instead of the simple one given above.

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* The differences are those of the function $\log L_x$.

Again, if the series consist of *six* terms—in which case there will be *four* terms in the second order of differences—the required effect may be produced by substituting for $\log L_x$ the series

$$\log L_0+(v-w),\ \log L_n-(v-w),\ \log L_{2n}+v,\ \log L_{3n}-v,$$
$$\log L_{4n}+(v+w),\ \log L_{5n}-(v+w),$$

v and w being determined from the equations

$$2w=\frac{AC-B^2}{A+2B+C}\text{ and }8v=\frac{A'C'-B'^2}{A'+2B'+C'},$$

where

$$A=\Delta_0^2+\Delta_n^2,\quad B=\Delta_n^2+\Delta_{2n}^2,\quad C=\Delta_{2n}^2+\Delta_{3n}^2,$$

and

$$A'=\Delta_0^3+4w,\quad B'=\Delta_n^3,\quad C'=\Delta_{2n}^3-4w.$$

Here again, by involving the corrections symmetrically, we obtain for the unknown quantities simple instead of complicated quadratic equations.

Applying these formulæ to the series at page 338, we have $2w=\cdot002651$, and $8v=\cdot007725$. The transformed series therefore becomes—

log L'_{14}=	·989980				
		—·049960			
log L'_{28}=	·940020		—·007274	log=$\bar{3}$·86177	
		—·057234			·58737
log L'_{42}=	·882786		—·028128	log=$\bar{2}$·44914	
		—·085362			·58733
log L'_{56}=	·797424		—·108760	log=$\bar{1}$·03647	
		—·194122			·58737
log L'_{70}=	·603302		—·420572	log=$\bar{1}$·62384	
		—·614694			
log L'_{84}=	$\bar{1}$·988608				

The logarithms of the third series, and their differences, show that the transformed series fulfils the required conditions.

I have now to show that this result has been attained without a greater alteration of the original series than is warranted by the probable errors of the latter. In the following table the first column contains the age, the second the natural numbers corresponding to the original series $\log L_x$, the third gives the decrement (deduced from the original data) of the year immediately following, while the fourth and fifth contain respectively the transformed series (denoted by L'_x)* and the amount by which it differs from the original series in the second column.

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* Hitherto the accent has been used to distinguish the "partial" from the "total" forces of mortality, but as we have now done with this branch of the subject, no confusion will be caused by using it to denote (as it will be used henceforth) the *corrected* values of the function to which it is applied.

x.	L_x.	D_x.	L'_x.	$L_x - L'_x$.
14	9780·0	41·4	9771·9	+ 8·1
28	8702·8	69·8	8710·0	− 7·2
42	7617·6	78·3	7634·6	− 17·0
56	6286·2	150·3	6272·3	+ 13·9
70	3990·3	220·3	4011·5	− 21·2
84	979·3	152·2	974·1	+ 5·2
		712·3		72·6

Comparing columns 3 and 5 together, term by term, we find that in one instance only (viz. at age 42) does the alteration made in the numbers living exceed one-fifth [·2171] of the corresponding yearly decrement; while from the sums of the same columns it appears that the *average* alteration is little more than one-tenth of the *average* decrement. We may, therefore, say the limit of the variation of the two series (cols. 2 and 4) is about one-fifth of a year.

If we turn now to the original table of the "Peerage Families," we shall find that the series showing the annual rate of mortality at successive ages exhibits considerable irregularity—in one instance (age 57) being one-half only of the mortality for the ages immediately preceding and following it. Now, I think few persons who have had much experience in the construction of mortality tables will venture to maintain that a table which contains such irregularities as these can be relied upon within anything like one-fifth of a year. I say *few* persons, because I am not sure but that some have an idea that the standard of accuracy is always to be found in the numbers determined from actual experience, however limited in extent, and however great may be the difference in the conditions under which the observations are made and those affecting the lives to which they are to be applied as the basis of computation.

Let us now examine the important observations conducted by Mr. A. G. Finlaison upon the lives generally known as the "Government Annuitants," male and female. Dividing the whole period of life comprised in these observations—viz., from one year upwards—as before, into intervals of 14 years, and neglecting the first, we have

$\log L_{15}=3{\cdot}98821$		
	$-{\cdot}06068$	
$\log L_{29}=3{\cdot}92753$		$-{\cdot}00633$
	$-{\cdot}06701$	
$\log L_{43}=3{\cdot}86052$		$-{\cdot}02670$
	$-{\cdot}09371$	
$\log L_{57}=3{\cdot}76681$		$-{\cdot}10771$
	$-{\cdot}20142$	
$\log L_{71}=3{\cdot}56539$		$-{\cdot}44978$
	$-{\cdot}65120$	
$\log L_{85}=2{\cdot}91419$		

Proceeding as before, we get the following transformed series ($\log L'_x$), which fulfils the given condition of a perfect geometrical progression in the four terms comprising the second order of differences :—

$$\begin{array}{lllll}
\log L'_{15}=3{\cdot}988254 & & & & \\
 & -{\cdot}060768 & & & \\
\log L'_{29}=3{\cdot}927486 & & -{\cdot}006357 & \log \bar{3}{\cdot}80325 & \\
 & -{\cdot}067125 & & & +{\cdot}61616 \\
\log L'_{43}=3{\cdot}860361 & & -{\cdot}026267 & \log \bar{2}{\cdot}41941 & \\
 & -{\cdot}093392 & & & +{\cdot}61621 \\
\log L'_{57}=3{\cdot}766969 & & -{\cdot}108548 & \log \bar{1}{\cdot}03562 & \\
 & -{\cdot}201940 & & & +{\cdot}61618 \\
\log L'_{71}=3{\cdot}565029 & & -{\cdot}448538 & \log \bar{1}{\cdot}65180 & \\
 & -{\cdot}650478 & & & \\
\log L'_{85}=2{\cdot}914551 & & & &
\end{array}$$

In this case the alterations introduced in the process of transformation are, in each of the six terms, perfectly insignificant, as the following table of the values of L_x, L'_x, and $L_x-L'_x$, will show :—

x.	L_x.	L'_x.	$L_x - L'_x$.
15	9732·2	9733·2	−1·0
29	8463·1	8462·3	+ ·8
43	7253·0	7250·4	+2·6
57	5845·3	5847·5	−2·2
71	3676·1	3673·1	+3·0
85	820·7	821·4	− ·7

It was originally my intention to extend the examination of the two preceding observations to the first 14 items of each series, which have been omitted in the foregoing comparison, with the view of deducing a formula which should be applicable to the whole period of life. But as such extension has no direct bearing upon the object which I have more immediately in view—viz., to establish a method for shortening the labour of forming tables of annuities on several lives, and for which object I find I shall have but little space to spare—I must postpone to a future opportunity the examination of so interesting and important a subject as the mortality of infancy and childhood.

In the records which we have next to examine, consisting of Mr. Hodgson's important observations on the clergy, we have necessarily no data for ages under 24. This circumstance has rendered it necessary to reduce the interval to 13 years.

$$\begin{array}{lll}
\log L_{24}=4{\cdot}00213 & & \\
 & -\ {\cdot}03057 & \\
\log L_{37}=3{\cdot}97156 & & -{\cdot}02059 \\
 & -\ {\cdot}05116 & \\
\log L_{50}=3{\cdot}92040 & & -{\cdot}07252 \\
 & -\ {\cdot}12368 & \\
\log L_{63}=3{\cdot}79672 & & -{\cdot}22938 \\
 & -\ {\cdot}35306 & \\
\log L_{76}=3{\cdot}44366 & & -{\cdot}74045 \\
 & -1{\cdot}09351 & \\
\log L_{89}=2{\cdot}35015 & &
\end{array}$$

The following are the adjusted series $\log L'_x$, and the comparative table of L_x and L'_x:—

$$\begin{array}{lllll}
\log L'_{24}=4\cdot001668 & & & & \\
 & -\cdot029646 & & & \\
\log L'_{37}=3\cdot972022 & & -\cdot022186 & \log=\bar{2}\cdot34608 & \\
 & -\cdot051832 & & & +\cdot50779 \\
\log L'_{50}=3\cdot920190 & & -\cdot071428 & \log=\bar{2}\cdot85387 & \\
 & -\cdot123260 & & & +\cdot50780 \\
\log L'_{63}=3\cdot796930 & & -\cdot229968 & \log=\bar{1}\cdot36167 & \\
 & -\cdot353228 & & & +\cdot50778 \\
\log L'_{76}=3\cdot443702 & & -\cdot740366 & \log=\bar{1}\cdot86945 & \\
 & -1\cdot093594 & & & \\
\log L'_{89}=2\cdot350108 & & & &
\end{array}$$

x.	L_x.	L'_x.	$L_x - L'_x$.
24	10049·2	10038·5	+ 10·7
37	9366·1	9376·1	− 10·0
50	8325·3	8321·3	+ 4·0
63	6262·1	6265·1	− 3·0
76	2777·5	2777·8	− ·3
89	223·9	223·9	·0

The numbers exposed to risk during the first five or six years of this table are so inconsiderable, that I should have preferred to exclude them, if it were not that the initial age is already somewhat advanced. In the next case, consisting of the observations known as the "Experience of the Seventeen Offices," Tables D(4) and E, I have neglected the ages under 20 and over 80, on account of the comparative insignificance of the numbers at risk at the excluded ages.

$$\begin{array}{lll}
\log L_{20}=3\cdot97023 & & \\
 & -\cdot04191 & \\
\log L_{32}=3\cdot92832 & & -\cdot00948 \\
 & -\cdot05139 & \\
\log L_{44}=3\cdot87693 & & -\cdot03332 \\
 & -\cdot08471 & \\
\log L_{56}=3\cdot79222 & & -\cdot09970 \\
 & -\cdot18441 & \\
\log L_{68}=3\cdot60781 & & -\cdot30823 \\
 & -\cdot49264 & \\
\log L_{80}=3\cdot11517 & &
\end{array}$$

By means of the formula previously used, we obtain the following values of $\log L'_x$:—

$$\begin{array}{lllll}
\log L'_{20}=3\cdot969914 & & & & \\
 & -\cdot041278 & & & \\
\log L'_{32}=3\cdot928636 & & -\cdot010580 & \log=\bar{2}\cdot02449 & \\
 & -\cdot051858 & & & +\cdot48803 \\
\log L'_{44}=3\cdot876778 & & -\cdot032548 & \log=\bar{2}\cdot51252 & \\
 & -\cdot084406 & & & +\cdot48810 \\
\log L'_{56}=3\cdot792372 & & -\cdot100144 & \log=\bar{1}\cdot00062 & \\
 & -\cdot184550 & & & +\cdot48809 \\
\log L'_{68}=3\cdot607822 & & -\cdot308114 & \log=\bar{1}\cdot48871 & \\
 & -\cdot492664 & & & \\
\log L'_{80}=3\cdot115158 & & & &
\end{array}$$

Taking out the natural numbers corresponding to the above, and comparing them with those of the original series, we have—

x.	L_x	L'_x	$L_x - L'_x$
20	9337·5	9330·7	+6·8
32	8478·5	8484·7	−6·2
44	7532·3	7529·7	+2·6
56	6197·5	6199·7	−2·2
68	4053·3	4053·4	− ·1
80	1303·7	1303·6	+ ·1

The differences (L_x—L'_x) are in this case also evidently unimportant; they do not in fact, in any instance, exceed one-tenth of the yearly decrement.

I have only one more instance to give—viz., the observations on the males of the Friendly Societies, by Mr. A. G. Finlaison. For the same reason as in the preceding case, the examination is restricted to the ages commencing with 20 and ending with 80. Here we have

$$\begin{array}{lll}
\log L_{20}=3{\cdot}79612 & & \\
 & -{\cdot}03947 & \\
\log L_{32}=3{\cdot}75665 & & -{\cdot}00922 \\
 & -{\cdot}04869 & \\
\log L_{44}=3{\cdot}70796 & & -{\cdot}02993 \\
 & -{\cdot}07862 & \\
\log L_{56}=3{\cdot}62934 & & -{\cdot}08296 \\
 & -{\cdot}16158 & \\
\log L_{68}=3{\cdot}46776 & & -{\cdot}24299 \\
 & -{\cdot}40457 & \\
\log L_{80}=3{\cdot}06319 & &
\end{array}$$

which is transformed into

$$\begin{array}{lllll}
\log L'_{20}=3{\cdot}795915 & & & & \\
 & -{\cdot}039053 & & & \\
\log L'_{32}=3{\cdot}756862 & & -{\cdot}010068 & \log=\bar{2}{\cdot}00294 & \\
 & -{\cdot}049121 & & & +{\cdot}46028 \\
\log L'_{44}=3{\cdot}707741 & & -{\cdot}029055 & \log=\bar{2}{\cdot}46322 & \\
 & -{\cdot}078176 & & & +{\cdot}46034 \\
\log L'_{56}=3{\cdot}629565 & & -{\cdot}083861 & \log=\bar{2}{\cdot}92356 & \\
 & -{\cdot}162037 & & & +{\cdot}46036 \\
\log L'_{68}=3{\cdot}467528 & & -{\cdot}242062 & \log=\bar{1}{\cdot}38392 & \\
 & -{\cdot}404099 & & & \\
\log L'_{80}=3{\cdot}063429 & & & &
\end{array}$$

The observations which we are now examining are chiefly valuable on account of the large number of the lives observed, and I have availed myself of this circumstance to compare the results of the original observations year by year with those of the formula. The first table contains the numbers living and yearly decrements (L_x and D_x), the second the rate of mortality (m_x), and the third the "expectation" or mean duration of life (E_x).

x.	L_x.	L'_x.	D'_x.	$L_x - L'_x$.	x.	L_x.	L'_x.	D'_x.	$L_x - L'_x$.
18	6333·0	6338·6	44·1	− 5·6	62	3659·0	3674·2	112·3	−15·2
19	6289·7	6294·5	44·0	− 4·8	63	3555·3	3561·9	116·8	− 6·6
20	6253·5	6250·5	44·0	+ 3·0	64	3439·2	3445·1	121·1	− 5·9
21	6201·8	6206·5	44·1	− 4·7	65	3326·7	3324·0	125·5	+ 2·7
22	6156·0	6162·4	44·2	− 6·4	66	3212·1	3198·5	129·9	+13·6
23	6108·4	6118·2	44·2	− 9·8	67	3069·2	3068·6	134·2	+ ·6
24	6063·0	6074·0	44·3	−11·0	68	2936·0	2934·4	138·1	+ 1·6
25	6020·5	6029·7	44·5	− 9·2	69	2801·6	2796·3	141·9	+ 5·3
26	5978·1	5985·2	44·7	− 7·1	70	2642·5	2654·4	145·3	−11·9
27	5937·9	5940·5	44·9	− 2·6	71	2497·0	2509·1	148·2	−12·1
28	5893·9	5895·6	45·2	− 1·7	72	2341·1	2360·9	150·6	−19·8
29	5849·7	5850·4	45·4	− ·7	73	2192·9	2210·3	152·3	−17·4
30	5804·3	5805·0	45·9	− ·7	74	2055·9	2058·0	153·3	− 2·1
31	5754·9	5759·1	46·2	− 4·2	75	1905·2	1904·7	153·4	+ ·5
32	5710·2	5712·9	46·7	− 2·7	76	1756·3	1751·3	152·5	+ 5·0
33	5670·6	5666·2	47·2	+ 4·4	77	1593·1	1598·8	150·6	− 5·7
34	5623·2	5619·0	47·8	+ 4·2	78	1459·6	1448·2	147·6	+11·4
35	5576·9	5571·2	48·4	+ 5·7	79	1285·6	1300·6	143·3	−15·0
36	5529·6	5522·8	49·2	+ 6·8	80	1156·6	1157·3	138·0	− ·7
37	5486·4	5473·6	50·0	+12·8	81	1000·0	1019·3	131·5	−19·3
38	5438·0	5423·6	50·8	+14·4	82	881·8	887·8	123·9	− 6·0
39	5385·2	5372·8	51·8	+12·4	83	..	763·9	115·3	
40	5331·4	5321·0	52·9	+10·4	84	..	648·6	105·8	
41	5270·4	5268·1	54·1	+ 2·3	85	..	542·8	95·6	
42	5217·2	5214·0	55·3	+ 3·2	86	..	447·2	85·2	
43	5162·9	5158·7	56·8	+ 4·2	87	..	362·0	74·4	
44	5104·6	5101·9	58·2	+ 2·7	88	..	287·6	63·8	
45	5044·3	5043·7	59·9	+ ·6	89	..	223·8	53·5	
46	4983·6	4983·8	61·7	− ·2	90	..	170·3	43·9	
47	4922·9	4922·1	63·5	+ ·8	91	..	126·4	35·1	
48	4857·0	4858·6	65·7	− 1·6	92	..	91·3	27·2	
49	4792·1	4792·9	67·9	− ·8	93	..	64·1	20·6	
50	4721·7	4725·0	70·2	− 3·3	94	..	43·5	14·9	
51	4651·6	4654·8	72·9	− 3·2	95	.	28·6	10·6	
52	4581·3	4581·9	75·5	− ·6	96	..	18·0	7·1	
53	4501·1	4506·4	78·5	− 5·3	97	..	10·9	4·6	
54	4427·2	4427·9	81·6	− ·7	98	..	6·3	2·8	
55	4345·2	4346·3	84·8	− 1·1	99	..	3·5	1·7	
56	4259·3	4261·5	88·3	− 2·2	100	..	1·8	·9	
57	4158·1	4173·2	91·9	−15·1	101	..	·9	·5	
58	4058·3	4081·3	95·7	−23·0	102	..	·4	·2	
59	3947·4	3985·6	99·7	−38·2	103	..	·2	·1	
60	3853·1	3885·9	103·7	−32·8	104	..	·1	·1	
61	3746·8	3782·2	108·0	−35·4	105	..	·0	·0	

x.	m_x.	m'_x.	x.	m_x.	m'_x.	x.	m_x.	m'_x.
18	·68	·692	41	1·01	1·026	64	3·27	3·516
19	·58	·698	42	1·04	1·061	65	3·45	3·777
20	·83	·704	43	1·13	1·100	66	4·45	4·061
21	·74	·711	44	1·18	1·142	67	4·34	4·371
22	·77	·716	45	1·20	1·187	68	4·58	4·708
23	·74	·723	46	1·22	1·237	69	5·68	5·074
24	·70	·730	47	1·34	1·292	70	5·51	5·473
25	·70	·739	48	1·34	1·351	71	6·24	5·908
26	·67	·746	49	1·47	1·416	72	6·33	6·379
27	·74	·755	50	1·48	1·487	73	6·24	6·892
28	·75	·766	51	1·51	1·565	74	7·33	7·448
29	·77	·778	52	1·75	1·649	75	7·82	8·053
30	·85	·789	53	1·64	1·741	76	9·29	8·708
31	·78	·803	54	1·85	1·842	77	8·38	9·419
32	·69	·817	55	1·98	1·952	78	11·92	10·189
33	·84	·833	56	2·38	2·072	79	10·03	11·023
34	·82	·851	57	2·40	2·203	80	13·54	11·925
35	·85	·870	58	2·73	2·345	81	11·82	12·900
36	·78	·890	59	2·39	2·501	82	20·36	13·953
37	·88	·913	60	2·76	2·670	83	15·93	15·088
38	·97	·938	61	2·34	2·855	84	12·50	16·311
39	1·00	·965	62	2·83	3·057	85	18·75	17·627
40	1·14	·994	63	3·27	3·277	86	14·63	19·040

x.	E_x.	E'_x.	$E_x - E'_x$.	x.	E_x.	E'_x.	$E_x - E'_x$.
18	44·79	44·78	+ ·01	50	22·00	22·04	− ·04
19	44·09	44·09	·00	51	21·32	21·35	− ·03
20	43·34	43·39	+ ·05	52	20·63	20·68	− ·05
21	42·69	42·69	·00	53	19·98	20·01	− ·03
22	42·00	41·99	+ ·01	54	19·30	19·34	− ·04
23	41·32	41·29	+ ·03	55	18·64	18·69	− ·05
24	40·62	40·58	+ ·04	56	18·00	18·04	− ·04
25	39·90	39·87	+ ·03	57	17·41	17·40	+ ·01
26	39·18	39·16	+ ·02	58	16·81	16·77	+ ·04
27	38·44	38·45	− ·01	59	16·26	16·15	+ ·11
28	37·72	37·73	− ·01	60	15·63	15·54	+ ·09
29	36·99	37·02	− ·03	61	15·05	14·94	+ ·11
30	36·28	36·30	− ·02	62	14·38	14·35	+ ·03
31	35·58	35·58	·00	63	13·77	13·77	·00
32	34·85	34·86	− ·01	64	13·21	13·20	+ ·01
33	34·09	34·14	− ·05	65	12·62	12·64	− ·02
34	33·36	33·42	− ·06	66	12·03	12·10	− ·07
35	32·63	32·69	− ·06	67	11·55	11·57	− ·02
36	31·90	31·97	− ·07	68	11·03	11·05	− ·02
37	31·15	31·25	− ·10	69	10·51	10·55	− ·04
38	30·41	30·53	− ·12	70	10·08	10·06	+ ·02
39	29·70	29·81	− ·11	71	9·61	9·59	+ ·02
40	28·99	29·09	− ·10	72	9·18	9·12	+ ·06
41	28·32	28·37	− ·05	73	8·73	8·68	+ ·05
42	27·60	27·66	− ·06	74	8·25	8·25	·00
43	26·87	26·94	− ·07	75	7·82	7·83	− ·01
44	26·17	26·23	− ·06	76	7·40	7·43	− ·03
45	25·47	25·52	− ·05	77	7·06	7·04	+ ·02
46	24·77	24·82	− ·05	78	6·61	6·67	− ·06
47	24·06	24·11	− ·05	79	6·37	6·31	+ ·06
48	23·37	23·42	− ·05	80	5·97	5·97	·00
49	22·68	22·72	− ·04				

I commend the three tables last given to the attention of those who entertain the notion that by adhering to the actual figures, or the "raw material" of the observations, we necessarily attain the highest possible degree of accuracy, and who look upon the process of adjustment as one of the fine arts, which imparts a pleasing regularity to our tables at the expense of truth. In no tables that I am acquainted with are the original data so little altered by the process of adjustment as in these, a circumstance attributable chiefly (no doubt) to the great number at risk at each age, and the exclusion of disturbing influences by the peculiar character of the observations. Let it not, then, be too hastily assumed (as I fear it sometimes is) that the more considerable alterations which we are compelled to make, when dealing with observations made under less favourable circumstances, are necessarily departures from truth. It may be difficult to determine the best mode of making these alterations, but we cannot get rid of the difficulty by evading it; and I have no hesitation in saying that the very worst course that could possibly be adopted is to pin our faith upon the crude results of observation, just as accident may have chanced to present them to us, and hoodwink ourselves into the belief that in so doing we are following the path indicated by experience. Let us have the *facts* by all means, but unless we also possess the power to interpret their meaning—to evolve the hidden laws of which they are the rude exponents—we shall, I am afraid, turn them but to a very poor account.

It may not be without interest to compare the decrements of life obtained by the theory of geometrical partial forces with those of De Moivre's celebrated hypothesis. In the preceding table we find the series is nearly stationary for a few years (or rather it begins with a slight decrease), and afterwards gradually increases with an augmenting velocity for a considerable period. The rapidity of its increase then slackens, and ultimately changes to a decrease at the age of 75, from which age the series rapidly diminishes until it becomes insignificant at about the age of 105.

The decrement at the climacteric age (75) is 153·4, while the smallest decrement in the preceding portion of the table is 44—(see ages 19 and 20). In this case the climacteric age is exactly the same as in Mr. Brown's adjustment of the "Clergy" observations; but in tables showing a heavier mortality in the preceding years it occurs somewhat earlier. Nevertheless, the general characteristics above enumerated will be found in all the tables of any authority.

Although subsequent observations have shown that De Moivre's attempt in the discovery of the law of mortality was made in the wrong direction, yet the idea was a happy and ingenious one, and the hypothesis by no means deserves the contumelious terms in which the late Mr. Morgan permitted himself to speak of it. De Moivre, we may be quite sure, was as fully alive as Mr. Morgan could be to the shortcomings of his hypothesis; but he was also equally alive (which the other was not) to the defects of the tables formed with the means of observation then existing. We may also safely assume that such a man as De Moivre, had he been living, would not have sanctioned the use of the Northampton table for granting annuities after the experience of the Equitable Society had proved the unfitness of that table for such a purpose; and until such experience had been obtained, his own "wretched hypothesis" would certainly have been as trustworthy a guide as the table which Mr. Morgan relied upon so implicitly—because (as he imagined) it was based upon *facts* and not upon *theory*.

I have now concluded my examination of the facts which have appeared to me sufficient fully to entitle Mr. Gompertz's hypothesis to the favourable reception it has met with from the scientific authorities of the day; and I will conclude this paper with a few remarks upon the formula which I have deduced from it to represent the numbers living at every age from the period immediately following childhood to the utmost extremity of life.

1. Although no two sets of observations will yield precisely the same value of q, yet the coincidence is, generally, sufficiently close to lead to the inference that this important constant differs from the others in the formula in being independent of the conditions which determine the mortality in different classes of individuals. In the mortality tables which I have constructed by the use of this formula, I have found that an average value of q may be used in all without materially affecting the result. This being the case, there remain but three constants to be determined in the formula for L_x, for which, of course, three terms of the function are sufficient. In other words, if we determine three values of L_x—say L_{15}, L_{50}, and L_{85}, or L_{15}, L_{55}, and L_{95}—from the original data, we have the means of constructing the entire series, which shall exhibit all the essential characteristics of the mortality table, as above described. Let any one endeavour to do this (with the same terms) by the ordinary rules of interpolation, and it will be speedily shown what a power it is that this discovery of Mr. Gompertz's has conferred upon us. For instance, if between the values of L_{15}, L_{50}, and L_{85}, of the Carlisle table, we interpolate by means of the method of

finite differences, we shall have a continually increasing decrement, which shortly after the age of 85 exceeds the number of survivors, so that the latter for all higher ages is negative. It is true that this absurdity may be avoided by *logarithmic* interpolation; but in that case we shall find that at the commencement of the series the differences are *positive*, so that the successive values of L_x (in that portion of the table) form an increasing series.

2. In a paper which I had the honour of reading during the last session of the Institute, I endeavoured to show that the principal defect of our tables consists not, as some imagine, in faulty methods of adjustment, or rather in the fault of over-adjustment, but in the difficulty, if not the impossibility, of securing an adequate correspondence between the bases of the table and the conditions which are known to affect the individual lives to which the table is to be applied. And it is evident that the path to improvement in this respect must lie in the minute subdivision of our observations, a necessity which will often compel us to rely upon data much more limited than it is desirable they should be. Now the use of a suitable formula with only three constants will enable us materially to lessen this disadvantage by the long series of years which is brought to bear upon the determination of the three terms required for the computation of the constants, a circumstance which is equivalent to a corresponding increase in the number of the lives observed.

3. We have seen that the expression for the number living at any given age in a normally-constituted increasing population—in which the yearly births as well as the yearly deaths (and consequently also the excess of the former over the latter) are proportional to the existing population—is of the same form as that representing the numbers living at the given age in a stationary population, and also the numbers living in a table of mortality. But the function $L_x v^x$ is also of the same form as the latter, for L_x contains a factor s^x which combines with that introduced by the interest of money. Hence it follows that to determine the number living between given ages in a population normally constituted—whether increasing or stationary—as well as the expectation of life and the value of annuities, the summation of a function of one form only (viz., $dg^{q^x}s^x$) is required.

4. The following transformation of the integral in question, $\int dg^{q^x}s^x dx$, is worthy of attention. $F_x = a + bq^x = -\frac{1}{L_x} \cdot \frac{dL_x}{dx}$, $\therefore\ -\log L_x = c + ax + \frac{b}{\log q} \cdot q^x$, $\therefore\ L_x = \varepsilon^{-c-ax-\frac{b}{\log q} \cdot q^x}$. Put

$v=\frac{b}{\log q}q^{x}$ and $n=-\frac{a}{\log q}$, whence $\varepsilon^{-ax}=\varepsilon^{\log q.nx}=q^{nx}$. $\therefore \mathrm{L}_x =\varepsilon^{-c}.q^{nx}.\varepsilon^{-v}=m\varepsilon^{-v}.q^{nx}$. Now the series L_x may be multiplied by any arbitrary quantity whatever. Multiply the last expression by $\left(\frac{b}{\log q}\right)^{n}.\frac{\log q}{m}$, by which the equation to the number living is changed to $\mathrm{L}_x=\varepsilon^{-v}\left(\frac{b}{\log q}\cdot q^{x}\right)^{n}.\log q=\varepsilon^{-v}.v^{n}.\log q$. But $v \log q =\frac{dv}{dx}$. Hence the integral in question becomes

$$\int \mathrm{L}_x dx=\int \varepsilon^{-v}.\,v^{n-1}dv,$$

the limits of the right-hand member being the values of v, corresponding to the limits of the left-hand member for values of x. The form thus obtained will be recognised as that of Euler's celebrated second integral—which, taken between the limits of 0 and ∞, is one of the most important of the definite integrals which have been tabulated; and we have seen that it is the form to which all summations connected with questions of population, the expectation of life, as well as the values of annuities, are reduced.

APPENDIX.

I.

Let the number of annual births (L_0) in a stationary population be equally distributed over the year, then will the number annually completing the age v (viz., L_v) be similarly distributed. Hence the number of completions taking place in any given fraction of a year Δx will be $\mathrm{L}_v.\Delta x$. And during any given portion of time Δx there will be

$\mathrm{L}_v\Delta x$	who complete the age		v,
$\mathrm{L}_{v+\Delta x}\Delta x$	„	„	$v+\Delta x$,
$\mathrm{L}_{v+2\Delta x}\Delta x$	„	„	$v+2\Delta x$,
. . .	. . .	. . .	. .
$\mathrm{L}_{v+\overline{p-1}.\Delta x}.\Delta x$	„	„	$v+\overline{p-1}.\Delta x$.

Taking the sum we have for the total number completing every Δxth of a year's age—between v and $v+p\Delta x$, or v and $v+n$—during every portion of time Δx—

$$(\mathrm{L}_v+\mathrm{L}_{v+\Delta x}+\mathrm{L}_{v+2\Delta x}+\ldots.+\mathrm{L}_{v+\overline{p-1}.\Delta x})\Delta x.$$

If Δx be decreased without limit, this becomes $\int_0^n \mathrm{L}_{v+x}dx$, which consequently denotes the total number constantly living between ages v and $v+n$.

Again, in any year $\mathrm{L}_v-\mathrm{L}_{v+1}$ will be the number dying between the ages v and $v+1$, $\mathrm{L}_{v+1}-\mathrm{L}_{v+2}$ between the ages $v+1$ and $v+2$, and so on. Hence the total number dying in a year between the ages v and $v+n$ is

$L_v - L_{v+n} = -\Delta L_v$. And the ratio of the deaths during one year between the given ages to the number constantly living between the same ages is

$$\frac{-\Delta L_v}{\int_0^n L_{v+x}dx}.$$

If the population, instead of being stationary, increase in a geometrical progression, and if in any given portion (Δx) of a year $L_v\Delta x$ complete the age v, then during the same period of time there will be

$$\begin{array}{lll}
L_v\Delta x & \text{who complete the age} & v, \\
L_{v+\Delta x}\Delta x.r^{-\Delta x} & \text{,,} \qquad \text{,,} & v+\Delta x, \\
\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot \\
L_{v+\overline{p-1}.\Delta x}\Delta x.r^{-(p-1)\Delta x} & \qquad\quad \text{,,} & v+\overline{p-1}.\Delta x;
\end{array}$$

and summing, we have for the total number completing every Δxth of a year of age, between v and $v+n$, during the given time Δx,

$$(L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{-2\Delta x} + \ldots\ldots + L_{v+\overline{p-1}.\Delta x}r^{-(p-1)\Delta x})\Delta x.$$

Hence decreasing Δx without limit, we have, for the number living between the ages v and $v+n$ at any given instant of time, $\int_0^n L_{v+x}r^{-x}dx$.

Suppose that the number completing each interval Δx of age do so altogether in the middle of the given period of time Δx. Then taking a succession of such intervals of time p in number, we shall have for the number living, in the middle of the

$$\begin{array}{lll}
\text{1st interval,} & & (L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{-2\Delta x} + \ldots\ldots + L_{v+\overline{p-1}\Delta x}r^{-(p-1).\Delta x})\Delta x; \\
\text{2nd} & \text{,,} & (L_v r^{\Delta x} + L_{v+\Delta x} + L_{v+2\Delta x}r^{-\Delta x} + \ldots\ldots + L_{v+\overline{p-1}\Delta x}r^{-(p-2)\Delta x})\Delta x; \\
\text{3rd} & \text{,,} & (L_v r^{2\Delta x} + L_{v+\Delta x}r^{\Delta x} + L_{v+2\Delta x} + \ldots\ldots + L_{v+\overline{p-1}\Delta x}r^{-(p-3)\Delta x})\Delta x; \\
\cdot & \cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
p\text{th} & \text{,,} & (L_v r^{(p-1)\Delta x} + L_{v+\Delta x}r^{(p-2)\Delta x} + L_{v+2\Delta x}r^{(p-3)\Delta x} + \ldots + L_{v+\overline{p-1}\Delta x})\Delta x.
\end{array}$$

Summing and dividing by p, we get

$$\frac{1+r^{\Delta x}+r^{2\Delta x}+\ldots\ldots+r^{(p-1).\Delta x}}{p}\times$$

$$(L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{-2\Delta x} + \ldots + L_{v+\overline{p-1}.\Delta x}r^{-(p-1)\Delta x})\Delta x;$$

and decreasing Δx without limit, this becomes $\left(n=p.\Delta x \text{ or } \frac{1}{p}=\frac{\Delta x}{n}\right)$

$$\frac{1}{n}\int_0^n r^x dx \times \int_0^n L_{v+x}r^{-x}dx.$$

But $\int_0^n r^x dx = \frac{r^n-1}{\log r}$. Hence we have, for the mean or average number of the population living between the ages v and $v+n$, during a period of n years, $\frac{r^n-1}{\log r^n}\int_0^n L_{v+x}r^{-x}dx$.

Again, the number of deaths during the same period will be, in the

1st interval, $-(\Delta L_v+\Delta L_{v+\Delta x}r^{-\Delta x}+\ldots\ldots\ldots+\Delta L_{v+\overline{p-1}\cdot\Delta x}r^{-\overline{p-1}\cdot\Delta x})\Delta x$;
2nd ,, $-(\Delta L_v r^{\Delta x}+\Delta L_{v+2\Delta x}+\ldots\ldots\ldots+\Delta L_{v+\overline{p-1}\cdot\Delta x}r^{-\overline{p-2}\cdot\Delta x})\Delta x$;
3rd ,, $-(\Delta L_v r^{2\Delta x}+\Delta L_{v+2\Delta x}r^{\Delta x}+\ldots\ldots+\Delta L_{v+\overline{p-1}\cdot\Delta x}r^{-\overline{p-3}\cdot\Delta x})\Delta x$;
. . . .
*p*th ,, $-(\Delta L_v r^{\overline{p-1}\cdot\Delta x}+\Delta L_{v+2\Delta x}r^{(p-2)\Delta x}+\ldots+\Delta L_{v+\overline{p-1}\Delta x})\Delta x$.

Summing and dividing by $p\Delta x$ or n, we have

$$-\frac{1+r^{\Delta x}+r^{2\Delta x}+\ldots+r^{(p-1)\Delta x}}{n}\times$$
$$(\Delta L_v+\Delta L_{v+\Delta x}r^{-\Delta x}+\ldots+\Delta L_{v+\overline{p-1}\cdot\Delta x}r^{-\overline{p-1}\Delta x})\Delta x;$$

and reducing Δx without limit, this becomes

$$-\frac{r^n-1}{\log r^n}\int_0^n\frac{dL_{v+x}}{dx}r^{-x}dx.$$

Dividing by the mean population, we have

$$\frac{-\int_0^n d^0}{\int_0^n L_{v+x}r^{-x}dx}.$$

which is the ratio of the average annual number of deaths during n years, between the ages v and $v+n$, to the mean population.

Instead of supposing the number living at age $v+x$ to be $L_{v+x}r^{-x}$, let us suppose it to be $L_{v+x}B_x$, where B_x may be any quantity whatever. We shall then have for the number living in the

1st interval, $(L_v+L_{v+\Delta x}.B_{\Delta x}+\ldots\ldots\ldots\ldots+L_{v+\overline{p-1}\cdot\Delta x}.B_{\overline{p-1}\cdot\Delta x})\Delta x$;
2nd ,, $(L_vB_{-\Delta x}+L_{v+\Delta x}+\ldots\ldots\ldots\ldots+L_{v+\overline{p-1}\Delta x}.B_{\overline{p-2}\cdot\Delta x})\Delta x$;
. . . . + . . .
*p*th ,, $(L_vB_{-\overline{p-1}\Delta x}+L_{v+\Delta x}.B_{-(p-2)\Delta x}+\ldots+L_{v+\overline{p-1}\cdot\Delta x})\Delta x$.

Hence, summing and dividing by p, and reducing Δx without limit, we have

$$\frac{1}{n}\int_0^n\left(L_{v+x}\int_{x-n}^x B_xdx\right)dx;$$

or, putting $\int_{x-n}^x B_xdx=C_x$, this becomes

$$\frac{1}{n}\int_0^n.L_{v+x}C_xdx,$$

which is the mean population, during n years, between the ages v and $v+n$.

If $B_x=r^{-x}$, $\int_{x-n}^x B_xdx$ (or C_x) $=\frac{r^n-1}{\log r}\cdot r^{-x}$, and the expression for the mean population becomes

$$\frac{r^n-1}{\log r^n}\int_0^n L_{v+x}r^{-x}dx,$$

which is the formula already arrived at.

The number of deaths, during the n years, in the preceding case will be, in the

1st interval, $-(\Delta L_v+\Delta L_{v+\Delta x}B_{\Delta x}+ \ldots\ldots\ldots\ldots +\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-1}.\Delta x})\Delta x$;
2nd ,, $-(\Delta L_v B_{-\Delta x}+\Delta L_{v+x}+\ldots\ldots\ldots\ldots +\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-2}.\Delta x})\Delta x$;
3rd ,, $-(\Delta L_v B_{-2\Delta x}+\Delta L_{v+\Delta x}B_{-\Delta x}+\ldots\ldots +\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-3}.\Delta x})\Delta x$;
. . . .
pth ,, $-(\Delta L_v B_{-\overline{p-1}.\Delta x}+\Delta L_{v+\Delta x}B_{-\overline{p-2}.\Delta x}+\ldots+\Delta L_{v+\overline{p-1}.\Delta x})\Delta x$.

Summing, dividing by n, and reducing Δx without limit, we have for the average annual deaths $-\frac{1}{n}\int_0^n \frac{dL_x}{dx}C_x dx$. And dividing by the expression for the mean population, we have

$$\frac{-\int_0^n \frac{dL_x}{dx}C_x dx}{\int_0^n L_{v+x}C_x dx}$$

for the ratio of the average annual deaths to the mean population during a period of n years between the ages v and $v+n$.

If $\Delta' L_x$ and $d' L_x$ be substituted for ΔL_x and dL_x respectively, we see that the preceding processes are also applicable to deaths resulting from one or more particular causes only.

II.

To expand the differential coefficient of y_x in terms of the central finite differences:—

$$(1.)\ n\frac{dy_x}{dx}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{1}{3}\Delta^3 y_x-\frac{1}{4}\Delta^4 y_x+\ldots.$$

$$=\Delta y_x-\frac{1}{2}(\Delta^2 y_{x-n}+\Delta^3 y_{x-n})+\frac{1}{3}(\Delta^3 y_{x-n}+\Delta^4 y_{x-n})-\frac{1}{4}(\Delta^4 y_{x-n}+\Delta^5 y_{x-n})+\ldots.$$

$$=(\Delta y_x-\frac{1}{2}\Delta^2 y_{x-n})-\frac{1}{6}\Delta^3 y_{x-n}+\frac{1}{12}\Delta^4 y_{x-n}-\ldots.$$

$$=\frac{\Delta y_x+\Delta y_{x-n}}{2}-\frac{1}{6}\Delta^3 y_{x-n}+\frac{1}{12}(\Delta^4 y_{x-2n}+\Delta^5 y_{x-2n})-\ldots.$$

$$=\frac{1}{2}(\Delta y_x+\Delta y_{x-n})-\frac{1}{6}(\Delta^3 y_{x-n}-\tfrac{1}{2}\Delta^4 y_{x-2n})+\ldots.$$

$$=\frac{1}{2}(\Delta y_x+y_{x-n})-\frac{1}{12}(\Delta^3 y_{x-n}+\Delta^3 y_{x-2n})+\ldots.$$

$$(2.)\ n\frac{dy_x}{dx}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{1}{3}\Delta^3 y_x-\frac{1}{4}\Delta^4 y_x+\frac{1}{5}\Delta^5 y_x-\frac{1}{6}\Delta^6 y_x+\ldots.$$

$$\Delta y_{x-n:2}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{3}{8}\Delta^3 y_x-\frac{5}{16}\Delta^4 y_x+\frac{35}{128}\Delta^5 y_x-\frac{63}{256}\Delta^6 y_x+\ldots.$$

$$n\frac{dy_x}{dx}=\Delta y_{x-n:2}-\frac{1}{24}\Delta^3 y_x+\frac{1}{16}\Delta^4 y_x-\frac{47}{640}\Delta^5 y_x+\frac{61}{768}\Delta^6 y_x-\dots\dots$$

$$-\frac{1}{24}\Delta^3 y_{x-3n:2}=\qquad -\frac{1}{24}\Delta^3 y_x+\frac{1}{16}\Delta^4 y_x-\frac{5}{64}\Delta^5 y_x+\frac{35}{384}\Delta^6 y_x-\dots\dots$$

$$n\frac{dy_x}{dx}=\Delta y_{x-n:2}-\frac{1}{24}\Delta^3 y_{x-3n:2}+\frac{3}{640}\Delta^5 y_x-\frac{3}{256}\Delta^6 y_x+\dots\dots$$

$$\frac{3}{640}\Delta^5 y_{x-5n:2}=\qquad +\frac{3}{640}\Delta^5 y_x-\frac{3}{256}\Delta^6 y_x+\dots\dots$$

$$\therefore\ n\frac{dy_x}{dx}=\Delta y_{x-n:2}-\frac{1}{24}\Delta^3 y_{x-3n:2}+\frac{3}{640}\Delta^5 y_{x-5n:2}-\dots\dots$$

By a process similar to the last, the following development of a definite integral may be obtained, viz.,

$$\int_x^{x+n} y_x dx=n\left\{y_{x+n:2}+\frac{1}{24}\Delta^2 y_{x-n:2}-\frac{17}{5760}\Delta^4 y_{x-3n:2}+\dots\dots\right\}$$

III.

PROBLEM.—To find an expression for the number living at any age when the law of mortality is such that the chances of living a year,* multiplied by a constant factor, form a series whose logarithms are in geometrical progression.

The above is the form in which the law of mortality, which I have endeavoured to establish in the preceding pages, was first propounded in a paper published in vol. viii. of this *Journal*. Shortly after its publication I was favoured by Mr. Sprague with the following elegant solution of the above problem, which I doubt not will be received with the interest which the communications of so excellent a writer invariably command. I may add that I believe it was the first attempt to give an analytical expression for the function in question.

" Let L_x be the number living at age x. Then the chances of living a " year at the ages x, $x+1$, $x+2$... are $\frac{L_{x+1}}{L_x}$, $\frac{L_{x+2}}{L_{x+1}}$, ... And by the " hypothesis, whatever the value of x, $\log\left(a\frac{L_{x+2}}{L_{x+1}}\right)=q\log\left(a\frac{L_{x+1}}{L_x}\right)$. Hence " $\log a+\log L_{x+2}-\log L_{x+1}=q\log a+q\log L_{x+1}-q\log L_x$. And $\log L_{x+2}$ " $-(q+1)\log L_{x+1}+q\log L_x=(q-1)\log a$.

" Now let $\log L_x=u_x$; then the last equation becomes

$$u_{x+2}-(q+1)u_{x+1}+q\,.\,u_x=(q-1)\log a.$$

" Separating the symbols of operation,

$$\{D^2-(q+1)D+q\}u_x=(q-1)\log a,$$

" or
$$(D-1)(D-q)u_x=(q-1)\log a.$$

---

* Of course any period whatever may be taken as the unit of time.

“ Integrating this equation of finite differences,

$$u_x=\log \mathrm{L}_x=c_1+c_2.q^x-x\log a,$$

“ and

$$\mathrm{L}_x=\varepsilon^{c_1+c_2q^x-x\log a}=\varepsilon^{c_1}.\,\varepsilon^{c_2q^x}\varepsilon^{-x\log a}.$$

“ Let now $\varepsilon^{c_1}=d$, $\varepsilon^{c_2}=g$, then we have

$$\mathrm{L}_x=\frac{dg^{q^x}}{a^x},$$

“ the expression required.

“ For the solution of the equation $(\mathrm{D}-1)(\mathrm{D}-q)u_x=(q-1)\log a$, we “ proceed as follows:—

“ Since $\mathrm{D}=1+\Delta$, we have

$$\Delta(\Delta+\overline{1-q})u_x=(q-1)\log a$$

$$\therefore\ u_x=c_1+c_2q^x+\frac{(q-1)\log a}{\Delta\{\Delta-(q-1\}}$$

$$=c_1+c_2q^x-\frac{\log a}{\Delta}\left(1-\frac{\Delta}{q-1}\right)^{-1}$$

$$=c_1+c_2q^x-\frac{1}{\Delta}\left(1+\frac{\Delta}{q-1}+\frac{\Delta^2}{(q-1)^2}+\,..\right)\log a$$

$$=c_1+c_2q^x-\left(\Sigma+\frac{1}{q-1}+\frac{\Delta}{(q-1)^2}+\,..\right)\log a$$

$$=c_1+c_2q^x-x\log a-\frac{\log a}{q-1}.$$

“ And changing the constants,

$$u_x=c_1+c_2q^x-x\ \mathit{log}\ a.$$

“ The expression for L_x may be got by a more gradual process, thus:—

“ Let p_x denote the chance of living a year at the age x, then by hypo- “ thesis, $\log(ap_{x+1})=q.\log(ap_x)$, whence

$$\log a+\log p_{x+1}=q\log a+q\log p_x,$$

“ and

$$\log p_{x+1}-q\log p_x=(q-1)\log a.$$

“ Then, putting $\log p_x=v_x$,

$$v_{x+1}-qv_x=(q-1)\log a$$

“ or

$$(\mathrm{D}-q)v_x=(q-1)\log a,$$

“ whence $v_x=\log p_x=cq^x-\log a$; and $p_x=\varepsilon^{cq^x-\log a}=\dfrac{\varepsilon^{cq^x}}{a}$; or, putting “ $\varepsilon^c=g$, $p_x=\dfrac{g^{q^x}}{a}$.

“ Then $\dfrac{\mathrm{L}_{x+1}}{\mathrm{L}_x}=\dfrac{g^{q^x}}{a}$, $\log \mathrm{L}_{x+1}-\log \mathrm{L}_x=q^x.\log g-\log a$,

$$u_{x+1}-u_x=q^x.\log g-\log a,\ \text{or}\ \Delta u_x=q^x\log g-\log a.$$

“ And, integrating,

$$u_x = \log \mathrm{L}_x = \frac{q^x \log g}{q-1} - x \log a + c$$

$$\mathrm{L}_x = \varepsilon^{\frac{q^x \log g}{q-1} - x \log a + c}$$

$$= \varepsilon^c . \varepsilon^{\frac{q^x \log g}{q-1}} . \varepsilon^{-x \log a}$$

$$\left.\begin{aligned} &= d\left(g^{\frac{1}{q-1}}\right)^{q^x} . \frac{1}{a^x} \\ &= \frac{d\gamma^{q^x}}{a^x} \end{aligned}\right\} \text{ (putting } \varepsilon^c = d,\ \gamma = g^{\frac{1}{q-1}}\text{).}$$

Mr. Sprague concludes with the following concise demonstration of the property of “ equal lives”:

$$p_{x,n} = \frac{\mathrm{L}_{x+n}}{\mathrm{L}_x} = \frac{dg^{q^{x+n}}}{a^{x+n}} \div \frac{dg^{q^x}}{a^x} = \frac{g^{q^x(q^n-1)}}{a^n}.$$

“ Similarly, $p_{y,n} = \dfrac{g^{q^y(q^n-1)}}{a^n}$;

$$\therefore \quad p_{(xy)n} = \frac{1}{a^{2n}} g^{(q^x+q^y)(q^n-1)}.$$

“ So also $p_{(zz)n} = \dfrac{1}{a^{2n}} g^{2q^z(q^n-1)}$.

“ Now, *whatever the value of n*, we shall have $p_{(xy)n} = p_{(zz)n}$, if z is found “ from the equation $2q^z = q^x + q^y$. This being the case, it is clear that any “ annuity, whether immediate, deferred, or temporary, on the joint lives x “ and y, is equal to the similar annuity on the joint lives z and z.”

IV.

The formulæ for the corrections to be applied to a series in order that the second order of differences may form a geometrical progression, are found as follows :—

Let the given series (B_x) consist of five terms; and, in order to produce the greatest effect upon the second differences with the least possible alteration of the original series, let the terms of the original series be alternately increased and diminished by the quantity p, the value of which is to be determined by the conditions of the problem.

$$\begin{array}{lll}
\mathrm{B}_0 + p & & \\
 & \Delta\mathrm{B}_0 - 2p & \\
\mathrm{B}_n - p & & \Delta^2\mathrm{B}_0 + 4p \\
 & \Delta\mathrm{B}_n + 2p & \\
\mathrm{B}_{2n} + p & & \Delta^2\mathrm{B}_n - 4p \\
 & \Delta\mathrm{B}_{2n} - 2p & \\
\mathrm{B}_{3n} - p & & \Delta^2\mathrm{B}_{2n} + 4p \\
 & \Delta\mathrm{B}_{3n} + 2p & \\
\mathrm{B}_{4n} + p & &
\end{array}$$

The three terms of the third series are to form a geometrical progression: therefore,

$$(\Delta^2B_0+4p)(\Delta^2B_{2n}+4p)=(\Delta^2B_n-4p)^2;$$

or $$\Delta^2B_0\times\Delta^2B_{2n}+4p(\Delta^2B_0+\Delta^2B_{2n})+(4p)^2=(\Delta^2B_n)^2-8p.\Delta^2B_n+(4p)^2$$

$$4p(\Delta^2B_0+2\Delta^2B_n+\Delta^2B_{2n})=(\Delta^2B_n)^2-\Delta^2B_0\times\Delta^2B_{2n}$$

$$\therefore\quad 4p=\frac{(\Delta^2B_n)^2-\Delta^2B_0\times\Delta^2B_{2n}}{\Delta^2B_0+2\Delta^2B_n+\Delta^2B_{2n}}.$$

Let the given series now consist of six terms. We shall, in this case, require two unknown quantities, which we shall find it convenient to introduce as follows:—

$$\begin{array}{lll}
B_0+(v-w) & & \\
 & \Delta B_0-2(v-w) & \\
B_n-(v-w) & & \Delta^2B_0+(4v-3w) \\
 & \Delta B_n+(2v-w) & \\
B_{2n}+v & & \Delta^2B_n-(4v-w) \\
 & \Delta B_{2n}-2v & \\
B_{3n}-v & & \Delta^2B_{2n}+(4v+w) \\
 & \Delta B_{3n}+(2v+w) & \\
B_{4n}+(v+w) & & \Delta^2B_{3n}-(4v+3w) \\
 & \Delta B_{4n}-2(v+w) & \\
B_{5n}-(v+w) & &
\end{array}$$

Before proceeding to determine the values of v and w, it will be necessary to prove the following theorem, viz.:—If we have any four quantities, a, b, c, and d, and if $b-a$, $c-b$, and $d-c$ are in geometrical progression, and likewise $b+a$, $c+b$, and $d+c$, then the four given terms a, b, c, and d, shall also form a geometrical progression.

By hypothesis, $(b-a)(d-c)=(c-b)^2$ and $(b+a)(d+c)=(c+b)^2$; or, multiplying out,

$$bd-ad-bc+ac=c^2-2bc+b^2,$$

and $$bd+ad+bc+ac=c^2+2bc+b^2,$$

whence, by adding and subtracting,

$$2bd+2ac=2c^2+2b^2;\text{ or } bd+ac=b^2+c^2.$$

And $$2ad+2bc=4bc;\text{ or } ad=bc.$$

From $bd+ac=b^2+c^2$ we get $\frac{a-c}{b}=\frac{b-d}{c}$; and from $ad=bc$ we have $\frac{a-c}{a}=\frac{b-d}{b}$. Dividing the first of these equations by the second, we get the equation $\frac{a}{b}=\frac{b}{c}$; $\therefore a:b::b:c$.

But from $ad=bc$, we see that $a:b::c:d$; consequently,

$$a:b::b:c::c:d,$$

and the four terms are in geometrical progression.

Applying this to the four terms of the second order of differences of the series

$$\begin{array}{l} \Delta^2B_0+(4v-3w)\\ \Delta^2B_n-(4v-w)\\ \Delta^2B_{2n}+(4v+w)\\ \Delta^2B_{3n}-(4v+3w) \end{array} \quad \begin{array}{l} \Delta^3B_0-8v+4w\\ \Delta^3B_n+8v\\ \Delta^3B_{2n}-8v-4w \end{array} \quad \left| \quad \begin{array}{l} \Delta^2B_0+\Delta^2B_n-2w\\ \Delta^2B_n+\Delta^2B_{2n}+2w\\ \Delta^2B_{2n}+\Delta^2B_{3n}-2w \end{array} \right.$$

We have to determine v and w, so that the three terms of each of the last two series shall be in geometrical progression. Putting $\Delta^2B_0+\Delta^2B_n=A$, $\Delta^2B_n+\Delta^2B_{2n}=B$, $\Delta^2B_{2n}+\Delta^2B_{3n}=C$; $\Delta^3B_0+4w=A'$, $\Delta^3B_n=B'$, and $\Delta^3B_{2n}-4w=C'$, and proceeding as in the determination of the value of $4p$, we shall find

$$2w=\frac{AC-B^2}{A+2B+C}, \text{ and } 8v=\frac{A'C'-B'^2}{A'+2B'+C'}.$$

V.

The following is a method of interpolating intermediate values in the series $\log L_0$, $\log L_n$, &c., which will be found convenient in constructing a mortality table according to the theory of geometrical partial forces.

Let Δ_x, $\Delta^2{}_x$, &c., represent the first and second differences of $\log L_x$ when the increment of x is unity, and ${}_n\Delta_x$, ${}_n\Delta^2{}_x$, &c., when the increment in question is n.

Then $${}_n\Delta_0=\Delta_0+(\Delta_0+\Delta^2_0)+(\Delta_0+\Delta^2_0+\Delta^2_1)+\ldots+(\Delta_0+\Delta^2_0+\Delta^2_1+\ldots+\Delta^2_{n-2})$$
$$=n\Delta_0+(n-1)\Delta^2_0+(n-2)\Delta^2_1+\ldots+\Delta^2_{n-2},$$

and

$${}_n\Delta_0-n\Delta_0=(n-1)\Delta^2_0+(n-2)\Delta^2_1+\ldots\ldots\ldots+\Delta^2_{n-2}$$
$$=\Delta^2_0\{(n-1)+(n-2)q+\ldots\ldots\ldots+2q^{n-3}+q^{n-2}\}. \qquad [1]$$

Put $\quad S=(n-1)+(n-2)q+(n-3)q^2+\ldots+2q^{n-3}+q^{n-2}$,

then $\quad Sq= \quad (n-1)q+(n-2)q^2+\ldots+3q^{n-3}+2q^{n-2}+q^{n-1}$,

and $S(q-1)=-n+1+q+q^2+\ldots\ldots+q^{n-3}+q^{n-2}+q^{n-1}=\dfrac{q^n-1}{q-1}-n.$

$$\therefore \qquad S=\frac{q^n-1}{(q-1)^2}-\frac{n}{q-1}.$$

Hence, substituting in [1], we have

$${}_n\Delta_0-n\Delta_0=\Delta^2_0\cdot\left\{\frac{q^n-1}{(q-1)^2}-\frac{n}{q-1}\right\}=\Delta^2_0\cdot\frac{(q^n-1)-n(q-1)}{(q-1)^2},$$

and

$$\frac{{}_n\Delta_0-n\Delta_0}{(q^n-1)-n(q-1)}=\frac{\Delta^2_0}{(q-1)^2}.$$

Again,

$${}_n\Delta^2_0={}_n\Delta_n-{}_n\Delta_0=n\Delta_n+\Delta^2_n\cdot\frac{(q^n-1)-n(q-1)}{(q-1)^2}-n\Delta_0-\Delta^2_0\cdot\frac{(q^n-1)-n(q-1)}{(q-1)^2}$$
$$=n(\Delta_n-\Delta_0)+(\Delta^2_n-\Delta^2_0)\frac{(q^n-1)-n(q-1)}{(q-1)^2}. \qquad [2]$$

But $\Delta_n-\Delta_0=\Delta_0^2+\Delta_1^2+\ldots+\Delta_{n-1}^2=\Delta_0^2(1+q+q^2+\ldots+q^{n-1})$

$$=\Delta_0^2\frac{q^n-1}{q-1}=\frac{\Delta_0^2}{(q-1)^2}\cdot(q^n-1)(q-1),$$

and $\Delta_n^2-\Delta_0^2=\Delta_0^2(q^n-1)$.

Hence, substituting in [2],

$$_n\Delta_0^2=\frac{\Delta_0^2}{(q-1)^2}\{n.(q^n-1)(q-1)+(q^n-1)^2-n.(q^n-1)(q-1)\}$$

$$=\frac{\Delta_0^2}{(q-1)^2}(q^n-1)^2.$$

$$\therefore\quad \frac{_n\Delta_0^2}{(q^n-1)^2}=\frac{\Delta_0^2}{(q-1)^2}=\frac{_n\Delta_0-n.\Delta_0}{(q^n-1)-n.(q-1)}.$$

By means of these equations the values of Δ_0 and Δ_0^2 may be determined from the values of $_n\Delta_0$ and $_n\Delta_0^2$. The successive terms of Δ_x^2 may then be calculated from the equation $\Delta_x^2=\Delta_0^2.q^x$.

On the Value of Apportionable Annuities; or of Annuities in which a proportionate part is payable up to the day of death. By THOMAS BOND SPRAGUE, M.A., *Actuary of the Equity and Law Life Assurance Society, and Vice-President of the Institute of Actuaries.*

IN former papers, I have investigated the value of an annuity on a single life, when payable half-yearly, quarterly, &c. It still remains to consider the values of annuities on the joint duration of two or more lives, or on the life of the last survivor of several lives, under similar circumstances. It will, however, be more useful to consider first the increase in the value of an annuity on a single life, when a proportionate part of the annuity is payable up to the day of death, instead of the annuity ceasing, as is generally assumed, with the payment that precedes the date of death.

The value of this increase will first be investigated on the supposition that the deaths in each year of age are uniformly distributed over the year.

Thus, the annuity being £1, payable yearly, and the age of the life on which the annuity depends being k, if death occur during the $(t+1)$th year—in the instant dx following the time $t+x$, then there becomes payable the sum £x. The deaths being uniformly distributed, the chance of this sum becoming payable, is

$$\frac{l_{k+t}-l_{k+t+1}}{l_k}\cdot dx=(p_{k,t}-p_{k,t+1})dx,$$

and the value of the chance of receiving it is therefore

$$xv^{x+t}(p_{k,t}-p_{k,t+1})dx.$$

The value of all such possible payments in the $(t+1)$th year will be found by integrating, and will be equal to

$$\int_0^1 xv^{x+t}(p_{k,t}-p_{k,t+1})dx$$

$$=v^t(p_{k,t}-p_{k,t+1})\int_0^1 xv^x dx.$$

Again, the value of all such possible payments throughout life will be found by summing this quantity with regard to t, giving t the values 0, 1, 2 . . . to the extremity of life.

Now $$\Sigma_t v^{t+1}(p_{k,t}-p_{k,t+1})=\text{A}_k.$$

Hence the value of the increase, which it will be convenient to call the "correction," becomes

$$\frac{\text{A}_k}{v}\int_0^1 xv^x dx, \text{ or } \text{A}_k\int_0^1 xv^{x-1}dx.$$

The value of the definite integral involved may be found as follows:—

$$xv^{x-1}=x(1+i)^{-\overline{x-1}}$$

$$=x-x(x-1)i+x(x-1)x\frac{i^2}{2}-x(x-1)x(x+1)\frac{i^3}{6}+\ldots.$$

(by the Binomial Theorem)

$$=x-(x^2-x)i+(x^3-x^2)\frac{i^2}{2}-(x^4-x^2)\frac{i^3}{6}+\ldots.$$

whence

$$\int xv^{x-1}dx=\text{C}+\frac{x^2}{2}-\left(\frac{x^3}{3}-\frac{x^2}{2}\right)i+\left(\frac{x^4}{4}-\frac{x^3}{3}\right)\frac{i^2}{2}-\left(\frac{x^5}{5}-\frac{x^3}{3}\right)\frac{i^3}{6}+\ldots(1)$$

and

$$\int_0^1 xv^{x-1}dx=\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{45}-\quad\ldots\ldots\ldots\ldots\quad(2)$$

Or, otherwise, integrating by parts,

$$\int xv^x dx=\frac{xv^x}{\log v}-\int\frac{v^x}{\log v}dx$$

$$=\frac{xv^x}{\log v}-\frac{v^x}{\log^2 v}+\text{C}$$

$$=\text{C}-\frac{xv^x}{\delta}-\frac{v^x}{\delta^2}\quad\ldots\ldots\ldots\ldots\quad(2^*)$$

$$\therefore\qquad\int_0^1 xv^x dx=\frac{1}{\delta^2}-\frac{v}{\delta}-\frac{v}{\delta^2},$$

and $$\int_0^1 xv^{x-1}dx = \frac{1}{v\delta^2} - \frac{1}{\delta} - \frac{1}{\delta^2} = \frac{i-\delta}{\delta^2},$$

where $\delta = \log_e(1+i) = -\log_e v$.

The two expressions we have found for $\int_0^1 xv^{x-1}dx$ may be shown as follows to be identical.

We have $1+i=e^\delta$

and $$i = e^\delta - 1 = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + \frac{\delta^4}{24} + \frac{\delta^5}{120} + \dots \quad (3)$$

whence $$\frac{i-\delta}{\delta^2} = \frac{1}{2} + \frac{\delta}{6} + \frac{\delta^2}{24} + \frac{\delta^3}{120} + \dots$$

But $$\delta = \log_e(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} - \dots\dots \quad (4)$$

and substituting this value for δ, we find

$$\frac{i-\delta}{\delta^2} = \frac{1}{2} + \frac{i}{6} - \frac{i^2}{24} + \frac{i^3}{45} - \dots$$

which agrees with (2).

Thus then the value of the correction is

$$A_k\left(\frac{1}{2} + \frac{i}{6} - \frac{i^2}{24} + \frac{i^3}{45} - \dots\right) \quad . \ . \ . \ . \ . \quad (5)$$

This expression agrees with the statement of Professor De Morgan, *Essay on Probabilities,* Appendix the second, p. xxii.— "When an annuity is granted upon condition that the executors of the party are to receive such a proportion of payment for the year in which the annuitant dies, as corresponds to the portion of the year during which he is alive, the addition to the value of the annuity is the present value of $\frac{1}{2} + \frac{i}{6}$ of a year's purchase, payable at the end of the year of death."

As a first approximation to the value of this correction, we may say that on the average £½ will be received in the middle of the year in which death takes place, and the value of the correction will therefore be nearly equal to

$$\frac{A_k}{2}\sqrt{1+i} \text{ or } A_k\left(\frac{1}{2} + \frac{i}{4} - \frac{i^2}{16} + \dots\right)$$

This result, it will be observed, is too large, and is not correct even in the term involving the first power of i.

Baily gives the value of the correction as simply $\frac{A_k}{2}$.

Griffith Davies has given (p. 336, cor. 1) a formula for the value of the correction which is equivalent to

$$\frac{i-\log(1+i)}{i^2}(1-ia_k)$$

$$=\frac{1+i}{i^2}\{i-\log(1+i)\}\mathrm{A}_k$$

$$=(1+i)\left(\frac{1}{2}-\frac{i}{3}+\frac{i^2}{4}-\frac{i^3}{5}+\ldots\right)\mathrm{A}_k$$

$$=\left(\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{20}-\ldots\right)\mathrm{A}_k,$$

which agrees with our formula as far as the term involving the first power of i—the remaining terms being different on account of the different supposition he has made as to the interest for fractions of a year.

If we next suppose the annuity to be payable half-yearly instead of yearly, a little consideration will show that the value of the possible payments in the $(t+1)$th year is

$$v^t(p_{k,t}-p_{k,t+1})(1+\sqrt{v})\int_0^{\frac{1}{2}} xv^x dx;$$

and the value of the correction is seen to be, reasoning as before,

$$\mathrm{A}_k\cdot\frac{1+\sqrt{v}}{v}\int_0^{\frac{1}{2}} xv^x dx$$

which, by (2*) $$=\mathrm{A}_k\frac{1+\sqrt{v}}{v}\left(\frac{1}{\delta^2}-\frac{\sqrt{v}}{2\delta}-\frac{\sqrt{v}}{\delta^2}\right)$$

$$=\mathrm{A}_k\frac{1+\sqrt{v}}{v}\left(\frac{1-\sqrt{v}}{\delta^2}-\frac{\sqrt{v}}{2\delta}\right)$$

$$=\mathrm{A}_k\left(\frac{1-v}{v\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}\right)$$

$$=\mathrm{A}_k\left(\frac{i}{\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}\right).$$

Now $\delta=-\log v$, and $v=e^{-\delta}$. Hence

$$v^{-\frac{1}{2}}=e^{\frac{\delta}{2}}=1+\frac{\delta}{2}+\frac{\delta^2}{4}\cdot\frac{1}{2}+\frac{\delta^3}{8}\cdot\frac{1}{6}+\frac{\delta^4}{16}\cdot\frac{1}{24}+\ldots.$$

$$\therefore\quad \frac{v^{-\frac{1}{2}}+1}{2\delta}=\frac{1}{\delta}+\frac{1}{4}+\frac{\delta}{16}+\frac{\delta^2}{96}+\frac{\delta^3}{768}+\ldots.$$

Also $$\frac{i}{\delta^2}=\frac{1}{\delta}+\frac{1}{2}+\frac{\delta}{6}+\frac{\delta^2}{24}+\frac{\delta^3}{120}+\ldots,\text{ by (3)},$$

$$\therefore \qquad \frac{i}{\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}=\frac{1}{4}+\frac{5}{48}\delta+\frac{\delta^2}{32}+\frac{9}{1280}\delta^3+\ldots.$$

and the value of the correction is

$$\mathrm{A}_k\left(\frac{1}{4}+\frac{5}{48}\delta+\frac{\delta^2}{32}+\frac{9}{1280}\delta^3+\ldots.\right)$$

$$=\mathrm{A}_k\left(\frac{1}{4}+\frac{5}{48}i-\frac{i^2}{48}+\frac{121}{11520}i^3-\ldots.\right) \quad . \quad . \quad (6)$$

This result may also be arrived at by finding $\int_0^{\frac{1}{2}} xv^x dx$ from the formula (1). Thus,

$$\int_0^{\frac{1}{2}} xv^x dx=\frac{1}{8}+\frac{i}{12}-\frac{5}{384}i^2+\frac{17}{2880}i^3-\ldots.$$

Again, $\quad 1+\sqrt{v}=1+(1+i)^{-\frac{1}{2}}$

$$=2-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ldots.$$

and multiplying these series together, we get the above expression (6) for the value of the correction.

Let the annuity now be supposed to be payable m times a year; and let each year, as the $(t+1)$th, be divided into m equal parts. Then if death occur in the $(s+1)$th part of the $(t+1)$th year, at the instant dx following the time $t+\frac{s}{m}+x$, there is payable the sum £x. The chance of death occurring in that instant on the supposition of uniform distribution of deaths is, as before, $(p_{k,t}-p_{k,t+1})dx$; and the value of the chance of receiving £x by such death is

$$xv^{t+\frac{s}{m}+x}(p_{k,t}-p_{k,t+1})dx.$$

The value of all the possible payments in the $(s+1)$th portion of the $(t+1)$th year will be found by integration, and is

$$v^{t+\frac{s}{m}}(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx.$$

The value of all the possible payments in the $(t+1)$th year will be found by summing with respect to s, giving it the values 0, 1 . . . $m-1$; and is therefore equal to

$$v^t(1+v^{\frac{1}{m}}+v^{\frac{2}{m}}+\ldots.+v^{\frac{m-1}{m}})(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx$$

$$=v^t\frac{1-v}{1-v^{\frac{1}{m}}}(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx,$$

Lastly, summing with respect to t, the total value of the correction is

$$\frac{1-v}{1-v^{\frac{1}{m}}}\cdot\frac{A_k}{v}\int_0^{\frac{1}{m}} xv^x dx=\frac{iA_k}{1-v^{\frac{1}{m}}}\int_0^{\frac{1}{m}} xv^x dx.$$

But, as we have already shown,

$$\int xv^x dx = C-\frac{xv^x}{\delta}-\frac{v^x}{\delta^2}$$

$$\therefore \qquad \int_0^{\frac{1}{m}} xv^x dx=\frac{1}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}-\frac{v^{\frac{1}{m}}}{\delta^2}$$

$$=\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta},$$

so that the value of the correction is

$$iA_k\left\{\frac{1}{\delta^2}-\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{v^{-\frac{1}{m}}-1}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{e^{\frac{\delta}{m}}-1}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{m\delta}\left(\frac{m}{\delta}-\frac{1}{2}+\frac{\delta}{12m}-\frac{1}{720}\frac{\delta^3}{m^3}+\dots\right)\right\}^*$$

$$=iA_k\left\{\frac{1}{2m\delta}-\frac{1}{12m^2}+\frac{\delta^2}{720m^4}-\dots\right\}$$

which becomes, since $\frac{i}{\delta}=1+\frac{\delta}{2}+\frac{\delta^2}{6}+\frac{\delta^3}{24}+\dots$

$$=1+\frac{i}{2}-\frac{i^2}{12}+\frac{i^3}{24}-\dots$$

$$A_k\left\{\frac{1}{2m}+\frac{i}{4m}\left(1-\frac{1}{3m}\right)-\frac{i^2}{24m}+\frac{i^3}{48m}\left(1+\frac{1}{15m^3}\right)-\dots\right\} \quad . \quad . \quad (7)$$

Making $m=1, 2, 4$, successively, the values of the corrections in the cases of annuities payable yearly, half-yearly, and quarterly, respectively, are found to be

$$A_k\left(\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{45}-\dots\right)$$

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* This step is obtained from the well known expansion of $\frac{t}{e^t-1}$.

$$A_k\left(\frac{1}{4}+\frac{5}{48}i-\frac{i^2}{48}+\frac{121}{11520}i^3-\ldots.\right)$$

$$A_k\left(\frac{1}{8}+\frac{11}{192}i-\frac{i^2}{96}+\frac{961}{184320}i^3-\ldots.\right)$$

the two first of which agree with the formulæ we have already found above.

The preceding formulæ are analagous to those obtained by Baily for the value of annuities payable half-yearly and quarterly, on the hypothesis of a uniform distribution of the deaths in each year. But we have seen in the case of those annuities that the results obtained by the use of this hypothesis are not to be relied on when accuracy is desired; and it will therefore be desirable to investigate the more accurate formulæ corresponding to Mr. Woolhouse's formula for the value of an annuity payable half-yearly.

Before doing this, however, it will be convenient to investigate the value of an ordinary annuity of which the first payment, instead of being made at the end of a year, is made sooner, say at the end of $\frac{1}{m}$ th part of a year.

In this investigation it is almost necessary, and is, at all events, highly desirable, to denote by a suitable symbol the value of the annuity in question. But when we seek for such a symbol, it becomes apparent that no convenient modification of the notation (a_k) for an ordinary annuity can be introduced; and that we must adopt a symbol to denote the value of an annuity-due, *i.e.*, of an annuity payable in advance. Mr. Scratchley has used the symbol a, and Mr. Filipowski, $\acute{a}$, for this purpose. I propose to adopt the symbol $\mathbf{a}$, as being more simple than the latter, and more distinctive than the former. Thus, then,

$$\mathbf{a}_k=1+a_k.$$

Next, suppose this annuity-due to be deferred n years, then $\mathbf{a}_{k\lceil\overline{n}}$ will denote an annuity of which the first payment is made at the end of n years; or

$$\mathbf{a}_{k\lceil\overline{n}}=a_{k|\overline{n-1}}.$$

Also, ${}_{\overline{t}|}\mathbf{a}_k$ will denote a temporary annuity of t payments, the first made at once; or

$${}_{\overline{t}|}\mathbf{a}_k=1+{}_{\overline{t-1}|}a_k.$$

So, again, $${}_{\overline{t}|}\mathbf{a}_{k\lceil\overline{n}}={}_{\overline{t}|}a_{k|\overline{n-1}}.$$

Making $n=0$, we see also that $a_{k|\overline{-1}}$ must be interpreted to mean $\mathbf{a}_k$.

Consistently with this notation, the value of an annuity of which the first payment is made at the end of the mth part of a year will be denoted by $\mathbf{a}_{k|\overline{\frac{1}{m}}}$.

We have for the value of this annuity

$$\mathbf{a}_{k|\overline{\frac{1}{m}}}=p_{k,\frac{1}{m}}v^{\frac{1}{m}}+p_{k,1+\frac{1}{m}}v^{1+\frac{1}{m}}+p_{k,2+\frac{1}{m}}v^{2+\frac{1}{m}}+\ldots.$$

$$=\frac{l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}+l_{k+1+\frac{1}{m}}v^{k+1+\frac{1}{m}}+l_{k+2+\frac{1}{m}}v^{k+2+\frac{1}{m}}+\ldots.}{l_kv^k};$$

or, putting u_x for l_xv^x,

$$\mathbf{a}_{k|\overline{\frac{1}{m}}}=\frac{u_{k+\frac{1}{m}}+u_{k+1+\frac{1}{m}}+u_{k+2+\frac{1}{m}}+\ldots.}{u_k}$$

Now substituting $\frac{1}{m}$ for x in the formula (8) for u_{k+x} given in my paper in the last Number of this *Journal*, p. 320, we have, very approximately,

$$u_{k+\frac{1}{m}}=u_k\left(1-\frac{3}{m^2}+\frac{2}{m^3}\right)+u_{k+1}\left(\frac{3}{m^2}-\frac{2}{m^3}\right)$$

$$+u'_k\left(\frac{1}{m}-\frac{2}{m^2}+\frac{1}{m^3}\right)+u'_{k+1}\left(-\frac{1}{m^2}+\frac{1}{m^3}\right)$$

$$=u_k-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)(u_k-u_{k+1})+\frac{u'_k}{m}\left(1-\frac{1}{m}\right)^2-\frac{u'_{k+1}}{m^2}\left(1-\frac{1}{m}\right)$$

So also,

$$u_{k+1+\frac{1}{m}}=u_{k+1}-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)(u_{k+1}-u_{k+2})+\frac{u'_{k+1}}{m}\left(1-\frac{1}{m}\right)^2-\frac{u'_{k+2}}{m^2}\left(1-\frac{1}{m}\right)$$

$$u_{k+2+\frac{1}{m}}=u_{k+2}-\&c.$$

$$\&c. \quad = \quad \&c.$$

Adding these equations together, we get

$$u_{k+\frac{1}{m}}+u_{k+1+\frac{1}{m}}+u_{k+2+\frac{1}{m}}+\ldots\ldots=\Sigma u_{k+\frac{1}{m}}$$

$$=u_k+u_{k+1}+u_{k+2}+\ldots\ldots-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)u_k$$

$$+\frac{1}{m}\left(1-\frac{1}{m}\right)^2(u'_k+u'_{k+1}+u'_{k+2}+\ldots.)$$

$$-\frac{1}{m^2}\left(1-\frac{1}{m}\right)(u'_{k+1}+u'_{k+2}+\ldots.)$$

$$=u_k+u_{k+1}+u_{k+2}+\ldots\ldots-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)u_k$$

$$+\frac{1}{m}\left(1-\frac{1}{m}\right)^2u'_k+\frac{1}{m}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)(u'_{k+1}+u'_{k+2}+\ldots)$$

But applying the formula (p. 207)

$$\frac{u_a}{2}+u_{a+1}+u_{a+2}+\ldots\ldots+\frac{u_b}{2}=\int_a^b u_x dx+\frac{1}{12}(u'_b-u'_a)$$
$$-\frac{1}{720}(u'''_b-u'''_a)$$
$$+\;\ldots\;.$$

we have

$$\frac{u'_k}{2}+u'_{k+1}+u'_{k+2}+\ldots.+\frac{u'_z}{2}=-u_k+\frac{u''_z-u''_k}{12}-\frac{u^{\mathrm{IV}}_z-u^{\mathrm{IV}}_k}{720}+\ldots.$$

—z being used (as at p. 317) to denote the extreme age in the table of mortality, which none of the lives actually attain.

The first term in the second member of this equation is obtained thus:

$$\int u'_x dx=u_x+\mathrm{C},$$

$$\therefore \qquad \int_k^z u'_x dx=u_z-u_k=-u_k, \quad \text{since } u_z=0.$$

The quantities u'_z, u''_z, u^{IV}_z. . . . are generally so small in comparison with the others that they may be neglected, and we have therefore very approximately

$$\Sigma u_{k+\frac{1}{m}}=\Sigma u_k-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)u_k+\frac{1}{m}\left(1-\frac{1}{m}\right)^2 u'_k$$
$$+\frac{1}{m}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(-u_k-\frac{u'_k}{2}-\frac{u''_k}{12}\right)$$
$$=\Sigma u_k-\frac{1}{m}u_k+\frac{1}{2m}\left(1-\frac{1}{m}\right)u'_k-\frac{1}{12m}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)u''_k.$$

Dividing both sides by u_k, we get

$$\mathbf{a}_{k\overline{|\frac{1}{m}}}=1+a_k-\frac{1}{m}+\frac{m-1}{2m^2}\cdot\frac{u'_k}{u_k}-\frac{(m-1)(m-2)}{12m^3}\cdot\frac{u''_k}{u_k}.$$

But since $u_k=l_k v^k$,

$$u'_k=l'_k v^k+\log v\, l_k v^k,$$

and $$\frac{u'_k}{u_k}=\frac{l'_k-\delta l_k}{l_k}=-\mu-\delta.$$

Also, $$u''_k=l''_k v^k-2\delta l'_k v^k+\delta^2 l_k v^k,$$

and $$\frac{u''_k}{u_k}=\frac{l''_k}{l_k}+2\mu\delta+\delta^2;$$

whence, by substitution,

$$\mathbf{a}_{k\overline{|\frac{1}{m}}}=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\left(\frac{l''_k}{l_k}+2\mu\delta+\delta^2\right)\;.\;.\;(8)$$

very approximately.

Or, adopting the notation I have employed in the last Number (p. 317),

$$\mathbf{a}_{k\overline{|\frac{1}{m}}}=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\quad .\;\;.\;\;.\quad (9)$$

The most important case is when $m=2$, or the first payment of the annuity is made at the end of six months. In that case,

$$\mathbf{a}_{k\overline{|\frac{1}{2}}}=a_k+\tfrac{1}{2}-\frac{\mu+\delta}{8}\quad .\;.\;.\;.\;.\;.\;.\quad (10)$$

the last term in the value of $\mathbf{a}_{k\overline{|\frac{1}{m}}}$ disappearing.

The universal practice of actuaries is to take the value of such an annuity as $a_k+\frac{1}{2}$; and we now see that that value is too large. The error, however, is seldom of much practical importance. Taking an average age, 45, and interest at three per cent., we have, according to the Experience table, $\frac{\mu+\delta}{8}=\cdot 0052$; and if we are valuing the liabilities of an Office in which the premiums amount to £100,000 a year, the error caused by using the common approximation will only amount to £520.

In order to test the accuracy of the above formula (9) we will apply it to find the value of an annuity payable m times a year. Writing $\frac{m}{r}$ for m, we have

$$\mathbf{a}_{k\overline{|\frac{r}{m}}}=a_k+\frac{m-r}{m}-\frac{r(m-r)}{2m^2}(\mu+\delta)-\frac{r(m-r)(m-2r)}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}.$$

Here making r equal to 1, 2, 3 . . . m, successively, we have, by addition,

$$\mathbf{a}_{k\overline{|\frac{1}{m}}}+\mathbf{a}_{k\overline{|\frac{2}{m}}}+\mathbf{a}_{k\overline{|\frac{3}{m}}}+\;.\;.\;.\;.\;.\;+\mathbf{a}_{k\overline{|1}}$$

$$=ma_k+m-\frac{1}{m}\Sigma r-\frac{\mu+\delta}{2m^2}\{m\Sigma r-\Sigma r^2\}$$

$$-\frac{\mathrm{D}''_k}{12m^3\mathrm{D}_k}\{m^2\Sigma r-3m\Sigma r^2+2\Sigma r^3\}.$$

But the first member of this equation is equal to $ma_k^{(m)}$.

Also $$\Sigma r=1+2+3+\ldots\ldots+m=\frac{m(m+1)}{2}$$

$$\Sigma r^2=1+2^2+3^3+\ldots+m^2=\frac{m(m+1)(2m+1)}{6}$$

$$\Sigma r^3=1+2^3+3^3+\ldots+m^3=\frac{m^2(m+1)^2}{4}$$

Hence we have

$$ma_k^{(m)}=ma_k+m-\frac{m+1}{2}-\frac{m+1}{4m}(\mu+\delta)\left\{m-\frac{2m+1}{3}\right\}$$

$$-\frac{\mathrm{D}''_k}{\mathrm{D}_k}\cdot\frac{m+1}{24m^2}\{m^2-m(2m+1)+m(m+1)\}$$

$$=ma_k+\frac{m-1}{2}-\frac{m^2-1}{12m}(\mu+\delta),$$

the other term disappearing;

and $$a_k^{(m)}=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta),$$

which is Mr. Woolhouse's formula. It will be noticed that in deducing this from our formula for $\mathbf{a}_{k\overline{\frac{r}{m}}|}$, no process of approximation has been employed; but if the one formula were strictly accurate, so also would the other be.

The above formula for $\mathbf{a}_{k\overline{\frac{1}{m}}|}$ needs no further confirmation. I subjoin, however, another demonstration, as illustrating a different mode of proceeding which may be found useful in other investigations.

We have, u_x being any function of x,

$$\frac{u_a}{2}+u_{a+h}+u_{a+2h}+\ldots\ldots+u_{b-h}+\frac{u_b}{2}$$

$$=\frac{1}{h}\int_a^b u_x dx+\frac{h}{12}(u'_b-u'_a)-\frac{h^3}{720}(u'''_b-u'''_a)+\ \ldots \quad (11)$$

Now put u_x equal to l_xv^x; and first let $a=k$, $h=1$, $b=z$, the extreme age in the table of mortality, which none of the lives actually attain, then the first member of (11) becomes $l_kv^k(\frac{1}{2}+a_k)$; and we get

$$l_kv^k(\tfrac{1}{2}+a_k)=\int_k^z u_x dx-\frac{1}{12}(u'_k-u'_z)+\frac{1}{720}(u'''_k-u'''_z)-\ldots\ldots$$

$$=\int_k^z u_x dx-\frac{u'_k}{12}+\frac{u'''}{720}-\ldots\ \ldots\ldots \quad (12)$$

approximately, the values u'_z, u'''_z, ... being very small.

Next, put $a=k+\frac{1}{m}$, $h=1$, b corresponding, as before, to the extremity of the mortality table; then we get

$$l_k v^k . \mathbf{a}_{k|\overline{\frac{1}{m}}}-\tfrac{1}{2}l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}=\int_{k+\frac{1}{m}}^{z}u_x dx-\frac{1}{12}u'_{k+\frac{1}{m}}+\frac{1}{720}u'''_{k+\frac{1}{m}}-\ldots \quad (13)$$

Lastly, put $a=k$, $h=\frac{1}{m}$, $b=a+h=k+\frac{1}{m}$, then the first member of (11) reduces to $\frac{u_a+u_b}{2}$ or $\frac{1}{2}(u_k+u_{k+\frac{1}{m}})$; and we have

$$\begin{aligned}\tfrac{1}{2}(l_k v^k+l_{k+\frac{1}{m}}v^{k+\frac{1}{m}})=m\int_{k}^{k+\frac{1}{m}}u_x dx&+\frac{1}{12m}(u'_{k+\frac{1}{m}}-u'_k)\\ &-\frac{1}{720m^3}(u'''_{k+\frac{1}{m}}-u'''_k)\\ &+\;.\;\;.\;\;.\;\;.\;\;.\end{aligned}$$

Dividing this by m, and adding it to (13), we get, since

$$\int_{k}^{k+\frac{1}{m}}u_x dx+\int_{k+\frac{1}{m}}^{z}u_x dx=\int_{k}^{z}u_x dx,$$

$$\begin{aligned}l_k v^k\left(\mathbf{a}_{k|\overline{\frac{1}{m}}}+\frac{1}{2m}\right)-\tfrac{1}{2}\left(1-\frac{1}{m}\right)l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}=\int_{k}^{z}u_x dx&\\ -\frac{1}{12m^2}\{(m^2-1)u'_{k+\frac{1}{m}}+u'_k\}&\\ +\frac{1}{720m^4}\{(m^4-1)u'''_{k+\frac{1}{m}}+u'''_k\}&\\ -\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.&\end{aligned}$$

Now subtracting (12) from this, we have

$$\begin{aligned}&l_k v^k\left\{\mathbf{a}_{k|\overline{\frac{1}{m}}}-a_k-\tfrac{1}{2}\left(1-\frac{1}{m}\right)\right\}-\tfrac{1}{2}\left(1-\frac{1}{m}\right)l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}\\ &\quad=\frac{m^2-1}{12m^2}(u'_k-u'_{k+\frac{1}{m}})-\frac{m^4-1}{720m^4}(u'''_k-u'''_{k+\frac{1}{m}})+\ldots.\end{aligned}$$

whence

$$\begin{aligned}\mathbf{a}_{k|\overline{\frac{1}{m}}}=a_k+\frac{m-1}{2m}\left(1+\frac{l_{k+\frac{1}{m}}}{l_k}\cdot v^{\frac{1}{m}}\right)&+\frac{m^2-1}{12m^2}\cdot\frac{u'_k-u'_{k+\frac{1}{m}}}{l_k v^k}\\ &-\frac{m^4-1}{720m^4}\cdot\frac{u'''_k-u'''_{k+\frac{1}{m}}}{l_k v^k}+\ldots.\end{aligned}$$

But since $u_x=l_x v^x$,

$$u'_x=l'_x v^x-\delta l_x v^x,$$

so that

$$u'_k=l'_k v^k-\delta l_k v^k,$$

and
$$u'_{k+\frac{1}{m}}=(l'_{k+\frac{1}{m}}-\delta l_{k+\frac{1}{m}})v^{k+\frac{1}{m}}$$
$$=\left\{l'_k+\frac{1}{m}l''_k+\ldots\ldots-\delta\left(l_k+\frac{l'_k}{m}+\frac{l''_k}{2m^2}+\ldots\right)\right\}v^{k+\frac{1}{m}}$$

(by Taylor's theorem)
$$=\left\{-\delta l_k+\left(1-\frac{\delta}{m}\right)l'_k+\frac{1}{m}\left(1-\frac{\delta}{2m}\right)l''_k+\ldots\right\}v^{k+\frac{1}{m}},$$

whence
$$u'_k-u'_{k+\frac{1}{m}}=-\delta l_k v^k(1-v^{\frac{1}{m}})+l'_k v^k\left\{1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right\}$$
$$-\frac{1}{m}\left(1-\frac{\delta}{2m}\right)l''_k v^{k+\frac{1}{m}}$$
$$+\ \ .\ \ .\ \ .\ \ .\ \ .$$

Also
$$\frac{l_{k+\frac{1}{m}}}{l_k}=\frac{1}{l_k}\left\{l_k+\frac{1}{m}l'_k+\frac{1}{2m^2}l''_k+\ldots\right\}$$
$$=1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}+\ldots.$$

We therefore have
$$\mathbf{a}_{k\overline{|\frac{1}{m}}}=a_k+\frac{m-1}{2m}\left\{1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)v^{\frac{1}{m}}\right\}$$
$$+\frac{m^2-1}{12m^2}\left\{-\delta(1-v^{\frac{1}{m}})-\mu\left(1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right)-\frac{1}{m}\left(1-\frac{\delta}{2m}\right)\frac{l''_k}{l_k}v^{\frac{1}{m}}\right\}$$
approximately.

Again, since $\delta=-\log v$, $v=e^{-\delta}$, and
$$v^{\frac{1}{m}}=e^{-\frac{\delta}{m}}=1-\frac{\delta}{m}+\frac{\delta^2}{2m^2}-\ldots.$$

we find, neglecting terms that involve δ^3, $\mu\delta^2$, and $l''_k\delta$,
$$\delta(1-v^{\frac{1}{m}})=\frac{\delta^2}{m},\text{ approximately;}$$
$$\mu\left(1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right)=\mu\left(\frac{\delta}{m}+\frac{\delta}{m}\right)=2\frac{\delta\mu}{m},\text{ approximately;}$$
$$\frac{1}{m}\left(1-\frac{\delta}{2m}\right)\frac{l''_k}{l_k}v^{\frac{1}{m}}=\frac{l''_k}{ml_k},\text{ approximately.}$$

Also
$$1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)v^{\frac{1}{m}}=1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)\left(1-\frac{\delta}{m}+\frac{\delta^2}{2m^2}\right)$$
$$=2-\frac{\mu+\delta}{m}+\frac{\delta^2}{2m^2}+\frac{\mu\delta}{m^2}+\frac{l''_k}{2m^2l_k},\text{ approximately;}$$

so that, substituting, we get finally

$$\mathbf{a}_{k\overline{|\frac{1}{m}}}=a_k+\frac{m-1}{2m}\left\{2-\frac{\mu+\delta}{m}+\frac{\delta^2}{2m^2}+\frac{\mu\delta}{m^2}+\frac{l''_k}{2m^2l_k}\right\}$$

$$+\frac{m^2-1}{12m^2}\left\{-\frac{\delta^2}{m}-\frac{2\delta\mu}{m}-\frac{l''_k}{ml_k}\right\}$$

$$=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\left(\delta^2+2\mu\delta+\frac{l''_k}{l_k}\right)$$

—as found above (8).

As a further test of the accuracy of this formula, apply it to the case of a perpetual annuity certain; and for this purpose make μ and l''_k both zero. Then a_k becomes equal to $\frac{1}{i}$; and the value of the annuity, as found from (8), becomes

$$\frac{1}{i}+\frac{m-1}{m}-\frac{m-1}{2m^2}\delta-\frac{(m-1)(m-2)}{12m^3}\delta^2\quad .\ .\ .\ (14)$$

But the value of the annuity certain in this case is

$$v^{\frac{1}{m}}+v^{1+\frac{1}{m}}+v^{2+\frac{1}{m}}+\ldots\text{ad inf.}$$

$$=\frac{v^{\frac{1}{m}}}{1-v}=\frac{1+i}{i}v^{\frac{1}{m}}=\frac{1}{i}v^{\frac{1}{m}-1}=\frac{e^{\left(1-\frac{1}{m}\right)\delta}}{i}=\frac{1}{i}\times e^{\frac{m-1}{m}.\delta}$$

$$=\frac{1}{i}\left\{1+\frac{m-1}{m}\delta+\frac{(m-1)^2}{m^2}\frac{\delta^2}{2}+\frac{(m-1)^3}{m^3}\cdot\frac{\delta^8}{6}+\ldots\right\}$$

$$=\frac{1}{i}+\frac{m-1}{m}\frac{\delta}{i}\left\{1+\frac{m-1}{m}\frac{\delta}{2}+\frac{(m-1)^2}{m^2.}\frac{\delta^2}{6}+\ldots\right\}$$

$$=\frac{1}{i}+\frac{m-1}{m}-\frac{m-1}{2m^2}\delta-\frac{(m-1)(m-2)}{12m^3}\delta^2+\ldots$$

—which is the same as found above (14).

For $\quad v=\frac{1}{1+i}=e^{-\delta}\quad\therefore\ 1+i=e^{\delta},\ $ and $\ i=e^{\delta}-1,$

$$\therefore\qquad\frac{\delta}{i}=\frac{\delta}{e^{\delta}-1}=1-\frac{\delta}{2}+\frac{\delta^2}{12}-\frac{\delta^4}{720}+\ldots\ldots\qquad(15)$$

this being the well known expansion by means of Bernouilli's numbers given in all treatises on the Calculus of Finite Differences.

We now return to the investigation of the increase in the value of an annuity, when upon the occurrence of death a further payment is made proportioned to the time which has elapsed since the last payment of the annuity. This increase will be called, as before, the "correction."

First, let the annuity be supposed to be payable yearly.

Then, the present age being k, if death occur in the $(t+1)$th year from the present time—say in the instant dx following the time x in that year; *i.e.*, at the time $t+x$ from the present time, there will be payable a sum £x. The chance of death occurring as supposed will be

$$\frac{l_{k+t+x}-l_{k+t+x+dx}}{l_k}=-\frac{l'_{k+t+x}}{l_k}\cdot dx,$$

and the value of £x to be received on the occurrence of such death will be

$$-xv^{t+x}\cdot\frac{l'_{k+t+x}}{l_k}\cdot dx \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (16)$$

and the value of the correction will be found by integrating this quantity with respect to x between the values 0 and 1; and then summing it with respect to t, giving t the values 0, 1, 2 .... to the extremity of life. Now we have, using the formula of interpolation employed above,

$$\begin{aligned} l_{k+t+x}=l_{k+t}&-(3x^2-2x^3)(l_{k+t}-l_{k+t+1}) \\ &+l'_{k+t}(x-2x^2+x^3)-l'_{k+t+1}(x^2-x^3) \quad . \;\; . \quad (17) \end{aligned}$$

Hence

$$l'_{k+t+x}=-6(x-x^2)(l_{k+t}-l_{k+t+1})+l'_{k+t}(1-4x+3x^2)-l'_{k+t+1}(2x-3x^2),$$

and substituting in (16) this value of l'_{k+t+x}, we have first to find the value of

$$\begin{aligned} \int_0^1\frac{v^{t+x}}{l_k}\Big\{6(x^2-x^3)(l_{k+t}-l_{k+t+1})&-l'_{k+t}(x-4x^2+3x^3) \\ &+l'_{k+t+1}(2x^2-3x^3)\Big\}dx \quad . \;\; . \quad (18) \end{aligned}$$

Now $\qquad \delta=-\log_e v \quad \therefore \; v=e^{-\delta}$

and $\qquad v^x=e^{-\delta x}=1-\delta x+\dfrac{\delta^2x^2}{2}-\dfrac{\delta^3x^3}{6}+\ldots.$

whence $\displaystyle\int x^nv^xdx=\int\Big(x^n-\delta x^{n+1}+\frac{\delta^2}{2}x^{n+2}-\frac{\delta^3}{6}x^{n+3}+\ldots.\Big)dx$

$$=\frac{x^{n+1}}{n+1}-\frac{\delta x^{n+2}}{n+2}+\frac{\delta^2x^{n+3}}{2(n+3)}-\frac{\delta^3x^{n+4}}{6(n+4)}+\ldots.+C,$$

and $\int_0^1 x^n v^x dx = \frac{1}{n+1} - \frac{\delta}{n+2} + \frac{\delta^2}{2(n+3)} - \frac{\delta^3}{6(n+4)} + \ldots.$

Here making $n=1, 2, 3$, successively, we get

$$\int_0^1 x\, v^x dx = \tfrac{1}{2} - \frac{\delta}{3} + \frac{\delta^2}{8} - \frac{\delta^3}{30} + \ldots.$$

$$\int_0^1 x^2 v^x dx = \tfrac{1}{3} - \frac{\delta}{4} + \frac{\delta^2}{10} - \frac{\delta^3}{36} + \ldots.$$

$$\int_0^1 x^3 v^x dx = \tfrac{1}{4} - \frac{\delta}{5} + \frac{\delta^2}{12} - \frac{\delta^3}{42} + \ldots.$$

Hence $\int_0^1 (x^2 - x^3) v^x dx = \tfrac{1}{12} - \frac{\delta}{20} + \frac{\delta^2}{60} - \frac{\delta^3}{252} + \ldots.$

$$\int_0^1 (x - 4x^2 + 3x^3) dx = -\tfrac{1}{12} + \frac{\delta}{15} - \frac{\delta^2}{40} + \frac{2}{315}\delta^3 - \ldots.$$

$$\int_0^1 (2x^2 - 3x^3) v^x dx = -\tfrac{1}{12} + \frac{\delta}{10} - \frac{\delta^2}{20} + \frac{\delta^3}{63} - \ldots.$$

and substituting these values in (18), we get the value of the correction in the $(t+1)$th year, equal to

$$-\frac{v^t}{l_k}\int_0^1 x v^x l'_{k+t+x} dx = \frac{v^t}{l_k}\left\{6\left(\tfrac{1}{12} - \frac{\delta}{20} + \frac{\delta^2}{60} - \frac{\delta^3}{252} + \ldots\right)(l_{k+t} - l_{k+t+1})\right.$$
$$+\left(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3 + \ldots\right) l'_{k+t}$$
$$\left. - \left(\tfrac{1}{12} - \frac{\delta}{10} + \frac{\delta^2}{20} - \frac{\delta^3}{63} + \ldots\right) l'_{k+t+1}\right\}.$$

Next summing with regard to t, giving it the values 0, 1, 2 . . . we get the total value of the correction, neglecting terms involving higher powers than δ^3,

$$\left.\begin{aligned} &= \left(\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42}\right)\frac{1}{l_k}\Sigma v^t (l_{k+t} - l_{k+t+1}) \\ &\quad + \left(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3\right)\frac{1}{l_k}\Sigma v^t l'_{k+t} \\ &\quad - \left(\tfrac{1}{12} - \frac{\delta}{10} + \frac{\delta^2}{20} - \frac{\delta^3}{63}\right)\frac{1}{l_k}\Sigma \iota^t l'_{k+t+1} \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

Now we notice that

$$\Sigma v^{t+1}\frac{l_{k+t} - l_{k+t+1}}{l_k} = \mathrm{A}_k$$

—the value of an assurance on the life k. But we shall find it more convenient to obtain an expression involving the annuity, a_k.

Thus $$\Sigma v^t \frac{l_{k+t}}{l_k} = 1 + a_k,$$

and $$\Sigma v^t \frac{l_{k+t+1}}{l_k} = \frac{1}{v} a_k = \left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)a_k,$$

so that

$$\left(\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42}\right)\frac{1}{l_k}\Sigma v^t(l_{k+t} - l_{k+t+1})$$

$$= \left(\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42}\right)\left\{1 + a_k - \left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)a_k\right\}$$

$$= \tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42} - \left(\frac{\delta}{2} - \frac{\delta^2}{20} + \frac{\delta^3}{30}\right)a_k, \text{ approximately.}$$

Again, $$\Sigma v^t l'_{k+t+1} = l'_{k+1} + v l'_{k+2} + v^2 l'_{k+3} + \ldots.$$

$$= \frac{1}{v}\{\Sigma v^t l'_{k+t} - l'_k\}$$

$$= \left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)\{\Sigma v^t l'_{k+t} - l'_k\}$$

whence

$$\left(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3\right)\frac{1}{l_k}\Sigma v^t l'_{k+t} - \left(\tfrac{1}{12} - \frac{\delta}{10} + \frac{\delta^2}{20} - \frac{\delta^3}{63}\right)\frac{1}{l_k}\Sigma v^t l'_{k+t+1}$$

$$= \left(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3\right)\frac{1}{l_k}\Sigma v^t l'_{k+t}$$

$$- \left(\tfrac{1}{12} - \frac{\delta}{60} - \frac{\delta^2}{120} - \frac{\delta^3}{504}\right)\frac{1}{l_k}(\Sigma v^t l'_{k+t} - l'_k)$$

$$= \left(-\frac{\delta}{20} + \frac{\delta^2}{30} - \frac{11}{2520}\delta^3\right)\Sigma\frac{v^t l'_{k+t}}{l_k}$$

$$- \left(\tfrac{1}{12} - \frac{\delta}{60} - \frac{\delta^2}{120} - \frac{\delta^3}{504}\right)\mu,$$

since $\mu = -\dfrac{l'_k}{l_k}$.

It only remains to evaluate $\Sigma v^t l'_{k+t}$

$$= l'_k + v l'_{k+1} + v^2 l'_{k+2} + \ldots.$$

Now putting $v^t l'_{k+t}$ for u_t in the equation

$$\frac{u_a}{2} + u_{a+1} + u_{a+2} + \ldots + \frac{u_b}{2} = \int_a^b u_t dt + \frac{u'_b - u'_a}{12}, \text{ nearly,}$$

and making $a=0$, $b=z-k$, we get

$$\Sigma v^t l'_{k+t} = \frac{l'_k}{2} + \int_0^{z-k} u_t dt - \frac{u'_0}{12}, \text{ very nearly,}$$

u'_{z-k} being so small that it may be neglected.

But
$$\int u_t dt = \int v^t l'_{k+t} dt$$
$$= v^t l_{k+t} + \delta \int v^t l_{k+t} dt$$
—integrating by parts,

and
$$\int_0^{s-k} u_t dt = -l_k + \delta \int_0^{s-k} v^t l_{k+t} dt$$
$$= -l_k + \delta l_k \left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12}\right), \text{ very nearly;}$$
since $\int_0^{s-k} \frac{v^t l_{k+t}}{l_k} dt$ is the value of an annuity payable momently.

Also
$$u'_t = -\delta v^t l'_{k+t} + v^t l''_{k+t},$$
and
$$u'_0 = -\delta l'_k + l''_k;$$
so that
$$\Sigma v^t l'_{k+t} = \frac{l'_k}{2} - l_k + \delta l_k \left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12}\right) + \frac{\delta l'_k - l''_k}{12},$$
and
$$\Sigma \frac{v^t l'_{k+t}}{l_k} = -\frac{\mu}{2} - 1 + \delta\left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12}\right) - \tfrac{1}{12}\left(\mu\delta + \frac{l''_k}{l_k}\right)$$
$$= \delta(a_k + \tfrac{1}{2}) - 1 - \frac{\mu}{2} - \tfrac{1}{12}\left(\delta^2 + 2\mu\delta + \frac{l''_k}{l_k}\right).$$

The value of the correction thus becomes, by successive substitutions in (19),
$$\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42} - \left(\frac{\delta}{2} - \frac{\delta^2}{20} + \frac{\delta^3}{30}\right)a_k$$
$$- \left(\frac{\delta}{20} - \frac{\delta^2}{30} + \frac{11}{2520}\delta^3\right)\left\{\delta(a_k + \tfrac{1}{2}) - 1 - \frac{\mu}{2} - \tfrac{1}{12}\left(\delta^2 + 2\mu\delta + \frac{l''_k}{l_k}\right)\right\}$$
$$- \left(\tfrac{1}{12} - \frac{\delta}{60} - \frac{\delta^2}{120} - \frac{\delta^3}{504}\right)\mu.$$

Multiplying out, and neglecting small quantities of the third order, involving δ^3, $\delta^2\mu$, or $\delta l''_k$, retaining however the term involving δ^3 in the coefficient of a_k, we get the value of the correction
$$\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \left(\frac{\delta}{2} - \frac{\delta^2}{20} + \frac{\delta^3}{30}\right)a_k$$
$$+ \frac{\delta}{20} - \frac{\delta^2}{30} - \left(\frac{\delta^2}{20} - \frac{\delta^3}{30}\right)a_k$$
$$- \frac{\delta^2}{40} + \frac{\mu\delta}{40} - \frac{\mu}{12} + \frac{\mu\delta}{60}$$
$$= \tfrac{1}{2} - \frac{\delta}{4} + \frac{\delta^2}{24} - \frac{\delta}{2}a_k - \frac{\mu}{12} + \frac{\mu\delta}{24}$$
$$= \tfrac{1}{2}(1 - \delta a_k) - \frac{\delta}{4} - \frac{\mu}{12} + \frac{\delta(\mu+\delta)}{24} \quad . \quad . \quad . \quad . \quad (20)$$

Hence, the value of an annuity payable yearly, with a proportionate part payable to the day of death, is

$$a_k+\tfrac{1}{2}-\frac{\delta}{4}+\frac{\delta^2}{24}-\frac{\delta}{2}a_k-\frac{\mu}{12}+\frac{\mu\delta}{24}$$

$$=(a_k+\tfrac{1}{2})\left(1-\frac{\delta}{2}\right)-\frac{\mu}{12}\left(1-\frac{\delta}{2}\right)+\frac{\delta^2}{24}$$

$$=\left(a_k+\tfrac{1}{2}-\frac{\mu}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{\delta^2}{24}$$

$$=\left(a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{\delta}{12} \quad . \quad . \quad . \quad . \quad (21)$$

This method will not give us the value of the annuity with greater accuracy; inasmuch as the fundamental equation (17) neglects small quantities of the third order.

When the annuity is payable, not yearly, but *m* times in the year, the value of the correction may be found as follows:—

Suppose the unit of time is taken to be $\frac{1}{m}$th part of a year, instead of one year, and that a payment of £1 is made at the expiration of each *m*-part. Then the amount of £1 at the end of one unit $\left(\frac{1}{m}\text{th of a year}\right)$ will be $(1+i)^{\frac{1}{m}}$; and we must substitute for δ in the equation (20), $\log(1+i)^{\frac{1}{m}}$; *i.e.* $\frac{1}{m}\log(1+i)$, or $\frac{\delta}{m}$. We must also substitute $ma_k^{(m)}$ for a_k. Lastly, with regard to μ; this quantity denoting the instantaneous mortality at the age *k*, referred to a unit of time (*i.e.* a year), we must substitute for μ, $\frac{\mu}{m}$, since $\frac{1}{m}$th of a year is now the unit of time. Making these substitutions, the formula (20) becomes

$$\tfrac{1}{2}\left(1-\frac{\delta}{m}\cdot ma_k^{(m)}\right)-\frac{\delta}{4m}-\frac{\mu}{12m}+\frac{\delta(\mu+\delta)}{24m^2}.$$

This applies to the value of an annuity of £*m* payable by *m* instalments in each year; and dividing by *m*, to obtain the proper formula for an annuity of £1, we get the value of the correction

$$\frac{1}{2m}(1-\delta a_k^{(m)})-\frac{\delta}{4m^2}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m^3}$$

$$\text{or } \frac{1}{2m}\left\{1-\delta\left[a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)\right]\right\}-\frac{\delta}{4m^2}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m^3}$$

$$=\frac{1}{2m}(1-\delta a_k)-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m} \quad . \quad . \quad (22)$$

Adding to this the value of $a_k^{(m)}$, we shall find that the value of an annuity payable m times a year, with a proportionate part to the day of death, is

$$\left(a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2} \quad . \quad . \quad . \quad (23)$$

If we put $m=1$, these expressions become the same as (20) and (21). If, again, we make m infinite, (23) reduces to

$$a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}$$

—the value of an annuity payable momently.

The preceding demonstrations being rather intricate, it will be satisfactory to show how the same results may be arrived at by a different process.

I will now take the case of an annuity payable m times a year. The year being, as before, supposed to be divided into m equal parts, let it be further supposed that each m-part is subdivided into n equal portions; and that if death occur in the rth subdivision of any m-part, the sum of $\frac{r}{mn}$ is payable at the end of that subdivision; then when n is supposed to become infinite, we get the case of an annuity payable m times a year with a proportionate part to the day of death. The value of the chance of receiving $\frac{r}{mn}$, as aforesaid, by the occurrence of death in the rth subdivision of the $(s+1)$th part of the $(t+1)$th year, is

$$\frac{r}{mn}\cdot v^{t+\frac{s}{m}+\frac{r}{mn}}\cdot\left\{p_{k,t+\frac{s}{m}+\frac{r-1}{mn}}-p_{k,t+\frac{s}{m}+\frac{r}{mn}}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

and we have now to sum this quantity with respect to t, s, and r.

It is immaterial which of these variables we deal with first. Summing first with regard to t, and giving it the values 0, 1, 2... to the extremity of life, the sum is

$$\frac{r}{mn}v^{\frac{s}{m}+\frac{r}{mn}}\left\{p_{k,\frac{s}{m}+\frac{r-1}{mn}}-p_{k,\frac{s}{m}+\frac{r}{mn}}\right.$$
$$+v\left(p_{k,1+\frac{s}{m}+\frac{r-1}{mn}}-p_{k,1+\frac{s}{m}+\frac{r}{mn}}\right)$$
$$+v^2\left(p_{k,2+\frac{s}{m}+\frac{r-1}{mn}}-p_{k,2+\frac{s}{m}+\frac{r}{mn}}\right)$$
$$+\cdot\cdot\cdot\cdot\cdot$$
$$\left.\vdots\qquad\qquad\qquad\qquad\right\}$$

$$=\frac{r}{mn}v^{\frac{1}{mn}}\left\{v^{\frac{s}{m}+\frac{r-1}{mn}}p_{k,\frac{s}{m}+\frac{r-1}{mn}}+v^{1+\frac{s}{m}+\frac{r-1}{mn}}p_{k,1+\frac{s}{m}+\frac{r-1}{mn}}+\dots\right\}$$
$$-\frac{r}{mn}\left\{v^{\frac{s}{m}+\frac{r}{mn}}p_{k,\frac{s}{m}+\frac{r}{mn}}+v^{1+\frac{s}{m}+\frac{r}{mn}}p_{k,1+\frac{s}{m}+\frac{r}{mn}}+\dots\right\}$$
$$=\frac{r}{mn}v^{\frac{1}{mn}}\mathbf{a}_{k\,|\overline{\frac{s}{m}+\frac{r-1}{mn}}}-\frac{r}{mn}\mathbf{a}_{k\,|\overline{\frac{s}{m}+\frac{r}{mn}}}\quad\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\quad(25)$$

—employing the notation I have explained in the earlier part of this paper, for the value of an annuity-due, deferred for a fraction of a year.

Next, summing with regard to s, giving it the values 0, 1, 2 . . . $m-1$, we have

$$\Sigma\mathbf{a}_{k\,|\overline{\frac{s}{m}+\frac{r-1}{mn}}}=\mathbf{a}_{k\,|\overline{\frac{r-1}{mn}}}+\mathbf{a}_{k\,|\overline{\frac{1}{m}+\frac{r-1}{mn}}}+\mathbf{a}_{k\,|\overline{\frac{2}{m}+\frac{r-1}{mn}}}+\dots+\mathbf{a}_{k\,|\overline{\frac{m-1}{m}+\frac{r-1}{mn}}}=m\mathbf{a}^{(m)}_{k\,|\overline{\frac{r-1}{mn}}},$$

and the sum of all the quantities (25) will be

$$\frac{r}{n}v^{\frac{1}{mn}}\mathbf{a}^{(m)}_{k\,|\overline{\frac{r-1}{mn}}}-\frac{r}{n}\mathbf{a}^{(m)}_{k\,|\overline{\frac{r}{mn}}}.$$

Here it will be noticed with reference to our notation that, $a^{(m)}_k$ denoting the value of an annuity of £1 payable by m instalments in each year; $\mathbf{a}^{(m)}_k$ denotes the value of a similar annuity, with the difference that the first payment is made at once instead of at the end of $\frac{1}{m}$th of a year; and $\mathbf{a}^{(m)}_{k\,|\overline{\tau}}$ denotes the value of a similar annuity deferred τ years, the first payment of which is therefore made at the end of the time τ, which may be either integral or fractional.

We have now to find the expressions for $\mathbf{a}^{(m)}_{k\,|\overline{\frac{r-1}{mn}}}$ and $\mathbf{a}^{(m)}_{k\,|\overline{\frac{r}{mn}}}$ analogous to that found above for $\mathbf{a}_{k\,|\overline{\frac{r}{m}}}$, viz.—

$$a_k+\frac{m-r}{m}-\frac{r(m-r)}{2m^2}(\mu+\delta)-\frac{r(m-r)(m-2r)}{12m^3}\frac{D'_k}{D_k}\quad\cdot\;\cdot\quad(26)$$

Now putting σ for $\frac{1}{m}$, we have

$$\mathbf{a}_{k\overline{|\tau}}=a_k+1-\tau-\frac{\tau(1-\tau)}{2}(\mu+\delta)-\frac{\tau(1-\tau)(1-2\tau)}{12}\frac{D''_k}{D_k}$$

$$\mathbf{a}_{k\overline{|\sigma+\tau}}=a_k+1-\sigma-\tau-\frac{\sigma+\tau}{2}(1-\sigma-\tau)(\mu+\delta)-\frac{\sigma+\tau}{12}(1-\sigma-\tau)(1-2\sigma-2\tau)\frac{D''_k}{D_k}$$

$$\mathbf{a}_{k\overline{|2\sigma+\tau}}=a_k+1-2\sigma-\tau-\frac{2\sigma+\tau}{2}(1-2\sigma-\tau)(\mu+\delta)-\frac{2\sigma+\tau}{12}(1-2\sigma-\tau)(1-4\sigma-2\tau)\frac{D''_k}{D_k}$$

$$\&c. \quad = \quad \&c.$$

$$\mathbf{a}_{k\overline{|(m-1)\sigma+\tau}}=a_k+1-(m-1)\sigma-\tau-\frac{(m-1)\sigma+\tau}{2}(1-\overline{m-1}.\sigma-\tau)(\mu+\delta)-\frac{(m-1)\sigma+\tau}{12}(1-\overline{m-1}.\sigma-\tau)(1-2.\overline{m-1}.\sigma-2\tau)\frac{D''_k}{D_k}.$$

Adding these together, the sum is the annuity we have denoted by $m\mathbf{a}^{(m)}_{k\overline{|\tau}}$; and we have

$$m\mathbf{a}^{(m)}_{k\overline{|\tau}}=ma_k+m-\frac{m(m-1)}{2}\sigma-m\tau-\frac{\mu+\delta}{2}\left\{\begin{array}{l}\tau+\sigma+\tau+2\sigma+\tau+\ldots\ldots\ldots\ldots+\overline{m-1}.\sigma+\tau\\ -\tau^2-(\sigma+\tau)^2-(2\sigma+\tau)^2-\ldots-(\overline{m-1}.\sigma+\tau)^2\end{array}\right\}$$

$$-\tfrac{1}{12}\frac{D''_k}{D_k}\Big\{\tau+\sigma+\tau+2\sigma+\tau+\ldots\ldots\ldots+\overline{m-1}.\sigma+\tau-3[\tau^2+(\sigma+\tau)^2+(2\sigma+\tau)^2+\ldots\ldots+(\overline{m-1}.\sigma+\tau)^2]$$

$$+2[\tau^3+(\sigma+\tau)^3+(2\sigma+\tau)^3+\ldots\ldots+(\overline{m-1}.\sigma+\tau)^3]\Big\}$$

$$=ma_k+m-\frac{m(m-1)}{2}\sigma-m\tau-\frac{\mu+\delta}{2}\left\{m\tau+\frac{m(m-1)}{2}\sigma-m\tau^2-m(m-1)\sigma\tau-\frac{(m-1)m(2m-1)}{6}\sigma^2\right\}$$

$$-\tfrac{1}{12}\frac{D'_k}{D_k}\Big\{m\tau+\frac{m(m-1)}{2}\sigma-3\left[m\tau^2+m(m-1)\sigma\tau+\frac{(m-1)m(2m-1)}{6}\sigma^2\right]$$

$$+2\left[m\tau^3+\tfrac{3}{2}m(m-1)\sigma\tau^2+\frac{3(m-1)m(2m-1)}{6}\sigma^2\tau+\frac{m^2(m-1)^2}{4}\sigma^3\right]\Big\}$$

which, since $\sigma=\frac{1}{m}$,

$$=ma_k+m-\frac{m-1}{2}-m\tau-\frac{\mu+\delta}{2}\left\{m\tau+\frac{m-1}{2}-m\tau^2-(m-1)\tau-\frac{(m-1)(2m-1)}{6m}\right\}$$

$$-\tfrac{1}{12}\frac{D''_k}{D_k}\left\{m\tau+\frac{m-1}{2}-3\left[m\tau^2+(m-1)\tau+\frac{(m-1)(2m-1)}{6m}\right]+2\left[m\tau^3+\tfrac{3}{2}(m-1)\tau^2+\frac{(m-1)(2m-1)}{2m}\tau+\frac{(m-1)^2}{4m}\right]\right\}$$

$$=ma_k+\frac{m+1}{2}-m\tau-\frac{\mu+\delta}{2}\left\{\frac{m^2-1}{6m}+\tau-m\tau^2\right\}-\tfrac{1}{12}\frac{D''_k}{D_k}\left\{\frac{\tau}{m}-3\tau^2+2m\tau^3\right\}.$$

Dividing by m,

$$\mathbf{a}^{(m)}_{k\overline{|\tau}}=a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{1}{12m^2}\frac{D''_k}{D_k}\cdot\tau(1-m\tau)(1-2m\tau)\quad\ldots\quad(27)$$

In order to test this result, first make $m=1$, or suppose the annuity payable yearly, then the equation reduces to

$$\mathbf{a}_{k\overline{|\tau}}=a_k+1-\tau-\frac{\mu+\delta}{2}\tau(1-\tau)-\tfrac{1}{12}\frac{D''_k}{D_k}\tau(1-\tau)(1-2\tau)\quad\ldots\ldots\ldots\quad(28)$$

—which agrees with (26) when we substitute $\frac{\tau}{m}$ for τ.

Next make $\tau=0$, then the equation becomes $\mathbf{a}^{(m)}_k=a_k+\frac{m+1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$.

But $\mathbf{a}^{(m)}_k=\frac{1}{m}+a^{(m)}_k$, so that this gives $a^{(m)}_k=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$

—Mr. Woolhouse's formula for the value of an annuity payable m times a year.

Again, make $\tau=\frac{1}{2m}$ in (27); then we get

$$\mathbf{a}^{(m)}_{k\overline{|\frac{1}{2m}}}=a_k+\frac{m+1}{2m}-\frac{1}{2m}-\frac{\mu+\delta}{12m^2}\left(m^2-1+3-\frac{3}{2}\right),$$

the other term disappearing,

$$=a_k+\tfrac{1}{2}-\frac{2m^2+1}{24m^2}(\mu+\delta),$$

which, making $m=1$, agrees with (10).

Here make $m=2$, then $\mathbf{a}^{(2)}_{k\overline{|\frac{1}{4}}}=a_k+\frac{1}{2}-\frac{3}{32}(\mu+\delta)$ . . . (29)

„ $m=4$, „ $\mathbf{a}^{(4)}_{k\overline{|\frac{1}{8}}}=a_k+\frac{1}{2}-\frac{11}{128}(\mu+\delta)$ . . . (30)

(*To be continued.*)

CORRESPONDENCE.

"EXPECTATION OF LIFE."

To the Editor.

SIR,—Is there anywhere to be found an exact and accurate definition of this function, called by some writers "expectation of life," by others "mean duration of life"? Mr. Peter Gray, in his valuable work entitled *Tables and Formulæ for the Computation of Life Contingencies*, has taken great pains to point out the errors of some preceding writers (see chapter v., pp. 59 to 72); and while treating the subject himself in a clear and satisfactory way, has been rather severe upon the inaccuracies of others. He therefore cannot, I think, reasonably object, if I direct attention to what seems to me a serious inaccuracy in his definition. He says, "By the mean duration of life at a specified age and according to a given table of mortality, is implied, the *average* number of years that, in the case of a single life, will be enjoyed by each individual of the specified age." WILL BE *enjoyed by* EACH *individual!!!* It is of course obvious that *each* individual cannot enjoy the *average* number of years, and that the preceding definition must be amended, by substituting the "average number of years which persons of the specified age, taken one with another, enjoy according to the given table of mortality."

Several writers object wholly to the use of the phrase "expectation of life"; and I notice in particular that Professor De Morgan in this *Journal*, vol. xii. p. 33, speaks of "the average life, or expectation, as it is *wrongly* called." Now, I can understand the objection made to the phrase, that it is somewhat misleading to persons who have not studied the subject accurately; though I do not admit that to be a sufficient reason for abandoning the use of an old phrase, which is convenient and well understood, as a technical term in the science of life contingencies. But what I do not understand is, why it should be said that this function is *wrongly called* "the expectation." It appears to me that the phrase is used in a sense strictly analogous to the sense it bears in other cases. Take the case of a number of persons drawing lots for a prize, what is meant by the "expectation of gain" possessed by any one of the persons? Or again, take the case of an Insurance Company with a given number of lives insured for specified sums, what is the "expectation of loss" in a particular year? It appears to me that the phrase "expectation of life" is strictly analogous to these other two phrases.

It is well known that the value of an annuity on a given life is less than that of an annuity certain for a term equal to the expectation of life; but I am only aware of one attempt at a strict proof of this proposition; that, namely, given by myself in the tenth volume of this *Journal*, p. 52. In that proof, however, I only established that the annuity is less than the annuity certain for the term of years indicated by the *nearest integer* to the number expressing the expectation of life. I now propose to complete the demonstration.

The expectation of life at a given age expresses the average number of payments of an annuity that would be made to persons of that age. Thus, then, in comparing the value of a life annuity with that of the annuity certain for a term equal to the expectation of life, we must take account, not only of the entire years included in that expectation, but also of the additional fraction of a year included therein. For example, if the expectation be equal to $n+\delta$, where n is an integer, and δ a fraction less than unity, we must compare with the value of the life annuity of £1 that of an annuity certain for n years with a further payment of δ at the end of the $(n+1)$th year. It will be noticed that the annuity being supposed, as usual, to cease with the payment at the end of the year preceding that in which death takes place, and not being continued up to the day of death, we therefore take for comparison the curtate and not the complete expectation of life.

Now, the age being x, the value of the life annuity is

$$\frac{l_{x+1}}{l_x}v+\frac{l_{x+2}}{l_x}v^2+\ .\ .\ .\ .\ +\frac{l_{x+s}}{l_x}v^s$$

$$=p_1v+p_2v^2+\ .\ .\ .\ .\ .\ +p_sv^s,\text{ suppose;}$$

and the value of the annuity certain is

$$v+v^2+v^3+\ .\ .\ .\ .\ +v^n+\delta v^{n+1}.$$

Also,
$$n+\delta=\frac{l_{x+1}+l_{x+2}+\ .\ .\ .\ .\ +l_{x+s}}{l_x}$$

$$=p_1+p_2+\ .\ .\ .\ .\ +p_s\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (1);$$

this equation expressing that if money be supposed to bear no interest, *i.e.*, $v=1$, then the value of the life annuity is equal to that of the annuity certain. Then the value of the life annuity will be greater or less than that of the annuity certain, according as

$$p_1v+p_2v^2+\ldots+p_sv^s > \text{ or } < v+v^2+\ldots+v^n+\delta v^{n+1} \quad . \; . \quad (2)$$

and therefore according as

$$p_{n+2}v^{n+2}+p_{n+3}v^{n+3}+\ldots+p_sv^s > \text{ or } <$$
$$(1-p_1)v+(1-p_2)v^2+\ldots+(1-p_n)v^n+(\delta-p_{n+1})v^{n+1}.$$

Now, since $v<1$, the first member of this inequality is less than

$$v^{n+2}(p_{n+2}+p_{n+3}+\ldots+p_s) \quad . \; . \; . \; . \quad (3)$$

and the second member is greater than

$$(1-p_1+1-p_2+\ldots+1-p_n+\delta-p_{n+1})v^{n+1}$$
$$>(n+\delta-p_1-p_2-\ldots-p_{n+1})v^{n+1}$$

or, from (1),

$$>(p_{n+2}+p_{n+3}+\ldots+p_s)v^{n+1}.$$

But this is greater than (3), and therefore returning to (2), we see that in all cases

$$p_1v+p_2v^2+\ldots+p_sv^s < v+v^2+\ldots+v^n+\delta v^{n+1},$$

or the life annuity is less than the annuity certain for the term equal to the expectation.

The reason for this result may be briefly explained as follows:—Comparing the life annuity and the annuity certain, and observing that the life annuity is on the average equivalent to an annuity certain of which the payment at the end of the first year is p_1, at the end of the second p_2, &c., we see that the total payments made in the two cases are equal, but in the life annuity, the payments are longer postponed. Thus taking the mth year, we have in the one case £1 payable at the end of the mth year, and in the other p_m payable at the end of the mth year, and $1-p_m$ payable at a time later than n years; and the value of the former is clearly the larger.

The reasoning of Jones on this point (Art. 127) appears to be not quite conclusive.

I am, Sir,
Your obedient servant,
T. B. SPRAGUE.

Equity and Law Life Assurance Society,
18, *Lincoln's Inn Fields,*
1*st March,* 1867.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

Third Ordinary Meeting. Session 1866–7.—*Monday, 28th January*, 1867.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected, viz.:—

Fellow.
Eugène Reboul.

Associates.
Carlton J. Lambert.
William Wellen Smith.
William Sutton, B.A.

The following was announced to be the result of the Examinations for 1866, viz.:—

Matriculation Examination.

Nine gentlemen sent in their names for this Examination, of whom two withdrew without giving in their papers, and five passed in the following order of merit, viz.:—

	Marks.		Marks.
Carlton J. Lambert	429	Henry Cowley	257
William Sutton	320	Walter Kember	253
James Chisholm	272		

Second Year's Examination.

Eleven gentlemen sent in their names for this Examination, of whom one withdrew, and two passed in the following order of merit, viz.:—

	Marks.
Robert W. Symes (with distinction)	604
Edwin Justican	416

The Examiners further reported that "several of the other gentlemen, although none had quite reached the required standard, had acquitted themselves creditably, and shown much industry and ability in their treatment of the questions."

Third Year's Examination.

Three gentlemen presented themselves for this Examination, and two passed in the following order of merit, viz.:—

Ralph Price Hardy (with distinction).
Charles Frederick Haycraft.

Mr. J. Hill Williams read a paper by Mr. Jardine Henry "On the real rate of mortality prevailing in England and Wales during the one hundred years from 1761 to 1861."

Thanks having been voted to Mr. Williams, the meeting adjourned to the 25th February, 1867.

Fourth Ordinary Meeting. Session 1866–7.—*Monday, 25th February*, 1867.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Joseph Druce.
Richard Lee.
Patrick Rattray.

Mr. Samuel Brown read a paper "On the mortality in the United States of America, as deduced from the last Census."

Thanks having been voted to Mr. Brown, the meeting adjourned to Monday, the 25th March, 1867.

Fifth Ordinary Meeting. Session 1866–7.—*Monday, 25th March,* 1867.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Robert James Spencer. | Charles Prentice.

Mr. Peter Gray read a paper "On the construction of tables by the method of differences." Part III.

Thanks having been voted to Mr. Gray, the meeting adjourned to Monday, the 29th April, 1867.

Sixth Ordinary Meeting. Session 1866–7.—*Monday, 29th April,* 1867.

The President in the chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman, duly nominated at the last ordinary meeting, was elected an Associate, viz.:—

Thomas Henry Griffith.

Mr. W. M. Makeham read a paper "On the law of mortality."

Thanks were voted to Mr. Makeham, and the meeting adjourned to Monday, 25th November, 1867.

The Twentieth Annual General Meeting, Saturday, 1st June, 1867.

SAMUEL BROWN, Esq., the President, in the Chair.

The Honorary Secretary, Mr. J. Hill Williams, read the minutes of the preceding ordinary meeting, and the following Reports:—

"The duty again devolves upon the Council of reporting to the members of the Institute as to its state and progress.

"The usual details will be found very fully set forth in the Assistant Secretary's letter, and in the annual account accompanying this Report.

"The income of the year appears to be £534. 16*s*., and the outgoing £370. 3*s*. 7*d*. The assets are estimated at £1,450.

"The following papers have been read during the Session:—

Nov. 26.—'On the value of annuities payable half-yearly, quarterly, &c.' By Mr. T. B. Sprague, M.A.

Dec. 17.—'On the limitation of risks.' (Part II.) By Mr. T. B. Sprague, M.A.

„ 'On the arrangement of the data furnished by certain Life Assurance Companies in Scotland, and on the formation of tables, &c., therefrom.' By Mr. James Meikle.

Jan. 28.—'On the real rate of mortality prevailing in England and Wales during the 100 years from 1761 to 1861.' By Mr. Jardine Henry.

Feb. 25.—'On the mortality in the United States of America as deduced from the last Census.' By Mr. Samuel Brown.

March 25.—'On the construction of tables by the method of differences.' By Mr. Peter Gray, F.R.A.S.

April 29.—'On the law of mortality.' By Mr. W. M. Makeham.

"The Committee appointed for the investigation of the rate of mortality amongst assured lives have made satisfactory progress in the classification of the data in their possession, and are continuing their labours. To the date of this Report the Offices contributing their experience are the Clerical and Medical, the Guardian, the Pelican, the London Assurance, the Union, the Metropolitan, the London and Provincial Law, the Equity and Law, the Northern, and the late Palladium Office; and the total number of lives on which observations have been registered is 75,080, of which 62,870 are male and 12,210 female.

"For the greater convenience of the members the Council have directed that a complete Catalogue of the Library be constructed, and considerable progress in the work has already been made by the gentleman to whom it has been entrusted.

"The Council have decided that a prize of ten guineas shall again be offered, to be competed for by associates who have passed their second year's examination. The subject of the Prize Essay to be 'Comparison of the values of policies as determined by the use of different tables of mortality.'

"As the members are aware, Mr. Jellicoe at the close of the Sessional Meetings intimated his wish to retire from the Presidentship, and also from the Editorship of the *Journal*, and the Council having, with great reluctance, accepted Mr. Jellicoe's resignation, it became necessary, in accordance with Art. XXVIII. of the Constitution and Laws of the Institute, to appoint a successor in the Presidency for the period intervening between the time of the vacancy occurring and that of the Annual General Meeting. A ballot having been taken with that object, at a special meeting of the Council held on the 7th May last, it appeared that Mr. Samuel Brown was unanimously elected. Mr. Brown has accordingly performed the duties of the office till the present time, and is now, as the members will perceive, recommended for re-election.

"The *Journal* will be placed, as the Council are glad to announce, under the editorship of Mr. Sprague after the issue of the forthcoming Number.

"SAMUEL BROWN,

"1*st June*, 1867." "*President.*

"*Institute of Actuaries*, 12, *St. James's Square*,
27th April, 1867.

"DEAR SIR,—I beg to send you herewith a copy of the audited Abstract of Receipts and Payments of the Institute of Actuaries for the financial year ended 31st March, 1867 (see p. 392), with a few remarks.

"RECEIPTS.

"The total receipts for the last two years were as follows:—1865-66, £474. 12*s*. 3*d*.; and 1866-67, £534. 16*s*. This large increase is partly owing to the arrears of some dividends being paid during the year.

"The sums received from subscriptions only in the last two years are as follows:—

1865–66.	Annual subscriptions . .	£464 2 0	
	Life " . .	..	
			£464 2 0
1866–67.	Annual subscriptions . .	£456 15 0	
	Life " . .	31 10 0	
			£488 5 0

"The bank balance at the end of the year was £251. 7*s*., whilst at the end of the previous year it was £270. 15*s*. 10*d*. This decrease is owing to the purchase of £200 3 per cent. Consols during the year.

"PAYMENTS.

"The excess of the annual subscriptions over the general expenses for the last two years is as follows, viz.:—

1865–66	£101 8 6
1866–67	£86 11 6

"The cost of printing the *Journal* was £66. 6*s*., as compared with £86. 19*s*. 6*d*. of the previous year. The payments for rent, ordinary meeting expenses, fire and light, postage, library, advertising examinations, and house and petty expenses, have all increased; whilst for stationery and sundry printing they have decreased.

"ASSETS.

"The investments on the 31st March last amounted to £823. 19*s*. 3*d*. sterling, and the balance at the bank and the petty cash balance to £259. 2*s*. The total available assets, therefore, at the end of the year, amounted to £1,083. 1*s*. 3*d*., as compared with £918. 8*s*. 10*d*. of the previous year.

"LIABILITIES.

"The following is an estimate of the liabilities of the Institute to the 30th September next, viz:—

	£		
Two quarters' salaries	£52	10	0
Rent	50	0	0
Journals—January, April and July (say)	65	0	0
Stationery and sundry printing (say)	10	0	0
Miscellaneous (say)	10	0	0
	£187	10	0

"MORTALITY EXPERIENCE INVESTIGATION FUND.

"The total amount hitherto contributed by various Insurance Offices towards this fund is £512. 5*s*.

"LIST OF MEMBERS.

"The total number of Members on the list on the 1st April, 1866, was 220; during the year 18 new members were elected—viz., 1 Fellow and 17 Associates; and the losses were 10, viz., 3 Fellows and 7 Associates; leaving on the list, on the 31st March, 1867, 228 members, 96 of whom were Fellows and 132 Associates. Two Associates have been transferred to the Fellows in the course of the year.

"I am, dear Sir, yours faithfully,

"FREDERICK GOVER,

"S. BROWN, Esq." "*Assistant Secretary.*

The PRESIDENT—" Gentlemen, in moving that this Report be adopted, entered on the minutes, and printed in the *Journal*, I will only make a few observations before inviting any remarks from other gentlemen; but I propose, before the close of the meeting, to review briefly the progress of the Institute to the present time. (Hear, hear.) I think, with regard to last year, every one must admit that we have fully maintained the reputation of the Institute, both by the papers which have been read and by the very interesting discussions to which they have given rise. (Hear, hear.) The papers by Mr. Sprague are of a practical character, at the same time that they lead the student to the consideration of some of the most difficult questions he will have to master in the course of his professional pursuits. Mr. Gray's papers 'On the construction of tables by the method of differences,' which are not yet finished, will, I believe, lead to some most interesting and important papers, which have yet to be written. But of all those which are likely to lead to further discussion of the highest importance is the paper by

Mr. Makeham, 'On the law of mortality.' (Hear, hear.) I will not say much about it at the present moment, as I shall allude to it a little more hereafter in the address which I shall have the honour of presenting to you. I am satisfied we are now in the right direction for extending our powers of calculation and also our inquiries on a subject which the greatest minds have hitherto been investigating without arriving at any very satisfactory conclusion up to the present time. In all other respects the progress of the Institute appears to have been highly encouraging, as regards the number of members and the increase of wealth, though we are a body which is not very wealthy, but which works in a satisfactory way with but little money. This is a source of congratulation, not only to the Council, but to every member of the Institute. (Cheers.) I shall be glad to hear any remarks, and will now move 'That the report be received and adopted, and printed in the *Journal.*'"

Mr. A. H. BAILEY seconded the motion.

Mr. SPRAGUE—"I rise with very great pleasure to support the adoption of this Report. I fully agree with you, Sir, that the Institute is making progress, upon which we may fairly congratulate ourselves. I am very glad indeed to hear that you propose to give us a sketch of the progress of the Institute from its origin—(hear, hear)—which, I am sure, will be received with great interest, especially by those who, like myself, have not been members from the commencement, and who will be glad to have, on the best authority, a history of the origin and early years of the Institute. (Hear, hear.) You will probably tell us that the Institute has not entirely carried out the high aims its founders entertained; but I think there is no doubt that at the present time it is doing a work of very great public benefit; that it is advancing the science of our subject and the interests of the assurance profession; and that, in fact, it is doing very valuable work in many ways. (Hear, hear.) The subject on which I wish particularly to address to the meeting a few remarks is that of the prize which the Council have again resolved to offer to the junior members of the Institute. This is not a new idea. We have had two prizes already offered, but not for several years past; and I am very glad now that the Council have resolved to resume the practice, and hope it will lead to some useful and interesting results. (Hear, hear.) One great benefit that I look to in giving a prize of this sort, the subject not being too difficult—and I think we must admit that the subject proposed at the present time is not too difficult—is, that it will enable the younger men to show what they can do. Having obtained a certificate of competency, as some of them have, they naturally expect the Institute should give them a helping hand whenever they apply for an appointment. The Council will be able to do that with much greater force, warmth, and zeal, if they can say of their own knowledge that 'this gentleman who is applying for an appointment is a competent man and understands the subject well.' If, therefore, those who are looking for promotion will send in an essay, they will afford everybody who reads it a means of judging whether they are really qualified actuaries. (Hear, hear.) It may be suggested perhaps that there is only one prize to be given, and therefore this will be a benefit to only one of those who compete; but I think that if we look a little further we shall see that it may benefit several of them. In the first place, there is nothing to prevent the Council, if they think fit, awarding a second prize, if there should be two deserving essays. I only throw that out as a suggestion which might possibly be carried out. Another thing: if the Council are of opinion that three or four essays are deserving of commendation and note, they have only to request the writers of them to send in their names, and they will receive that mark of public approval which could not fail to be very serviceable to them."

Mr. BRABROOK.—"I rise to interpose before the resolution is put, because I think some one outside of the Council should express how exceedingly satisfactory the Report is, and how greatly it reflects credit on the Council. I rise also for the purpose of urging on the Council a suggestion which may have its value; and I do it to-day because this is the only occasion when the ordinary members are able to bring forward suggestions such as that which I propose to make. It has frequently occurred to me that we place ourselves at a disadvantage by publishing a quarterly journal of a fixed size. I think that if we follow the universal custom of all other learned Societies, with the exception of the Statistical Society, of publishing only those papers which are read before the Institute, and on which we are willing to stake our reputation, or on which responsible authors are willing to stake their reputation, it would be more advantageous. (Hear, hear; and No, no.) It would give the Society a higher standing in the estimation of foreign students and corresponding members who are constantly reading our *Journal*, and it would also give more time for the revision of papers by the editor, and enable the writers themselves to reduce their language to the most precise and logical accuracy, making the whole thing infinitely more creditable to the Society. (Hear, hear, and dissent.) I do not for one moment desire to say that the past conduct of the *Journal* has been anything but the most admirable possible. (Hear. hear.) Anybody who knows the editor must be aware that he has brought to that, as he brought to everything else, the most consummate tact and excellent spirit—(hear, hear)—and there is nobody who has ever corresponded with him but must feel how courteous, kind, and good has been his whole manner of editing the *Journal.* (Cheers.) But it seems to me that to bind yourselves to produce so many pages of matter every quarter is putting yourselves at a disadvantage. I think that you should produce just so much of what is brought before you as is calculated to further your reputation before the world. The only point, as it appears to me, which might be urged on the other hand is this—that our *Journal* is the receptacle of a good deal of correspondence of various kinds, which has its value. With respect to a portion of your correspondence—those excellent communications, for instance, which we get from our foreign members, Dr. Wiegand and others—would, of course, come as communications to the Institute, and appear in their proper place. In reference to this matter it is to be recollected that since the *Journal* was started there have been some valuable insurance newspapers and other works published. For instance, the *Insurance Record* is a most excellently conducted periodical—(hear, near)—affording an opportunity for correspondence on any subject which it may be necessary to debate. (Hear, hear.) When the points are all brought out, a member of the Institute who may be considered best qualified to write on the particular subject discussed should prepare a paper to be read before the Institute and inserted in the *Journal.* But that the Institute should put forward in its own official pages points which are not settled, is not, I think, worthy of its reputation. It is not necessary to trouble you with further observations on the subject. I am sure the suggestion will receive the kind consideration of the Council, and be taken for what it is worth. (Hear, hear.) If adopted, it will not diminish the responsibility of our new editor; and, if not adopted, I shall feel that some good reason can be urged against it, and shall be content to wait till the time comes, as I believe sooner or later it will come, when it will be adopted by the Council." (Hear, hear.)

The Report was then adopted unanimously.

The President mentioned that the auditors—Messrs. G. W. Berridge, B. Newbatt, and R. P. Hardy—had expressed their willingness to act for the ensuing year; and they were accordingly reappointed.

Two gentlemen were then selected to act as scrutineers in the election of the Council and officers of the Institute, and in the meantime

Mr. Hodge rose and said—" Mr. President, while the gentlemen who have so kindly undertaken the office of scrutineers are engaged in examining the list, perhaps you will allow me to intrude upon the meeting for a short time with an object which I think ought not to be lost sight of in such a meeting as this, and that is to propose a resolution in recognition of the services of the gentleman who has retired from the office of President. (Hear, hear.) It is not necessary that I should detain you long upon this subject, because I am sure that many of you—I hope all of you—will in a very short time have an opportunity of expressing your thanks in a more cordial and hearty way than is usual in such a meeting as this, and that you will also hear from another member of the Institute a more adequate appreciation of the services of the gentleman I allude to than I could possibly hope to give. At the same time, what we are going to do presently is a personal recognition on our part of the feeling of the members towards Mr. Jellicoe; but I think it would be quite contrary to our duty if we were to allow such a meeting as this to pass without placing on record, and that in the most emphatic manner, our high sense of the value of his services to the Institute of Actuaries. (Hear, hear.) If we consider the interest he took in its formation, the strenuous exertions he made for its establishment, and the long time he has held the office of President with so much efficiency and advantage to the Institute, I am sure we shall all cordially join in supporting this resolution. (Hear, hear.) I have so framed the motion as to make it obvious upon the face of it that the retirement of Mr. Jellicoe was his own act, and did not arise from any feeling on the part of the Institute. And, moreover, I have thought the meeting would like to offer a recognition of his services, both during the time he held the office of President, and also previously to that period; for it will be recollected that, although Mr. Jellicoe has only been President for seven years, yet for a long time previously, in consequence of the illness of the former President, and his inability to discharge the duties of the office, the principal duties fell upon Mr. Jellicoe. (Hear, hear.) How those duties were performed it is not necessary for me to say. I am sure every member of the Institute must have felt the regret which was expressed by the Council when Mr. Jellicoe retired from the office. (Hear, hear.) And in saying that, no one present will believe—and you, Sir, will be the last person to believe—that I intend to offer anything derogatory of your high claims, your fitness for holding that office, and the many valuable qualities you possess for filling it. (Hear, hear.) Nobody, I am sure, recognizes those claims and qualifications more highly than I do; but at the same time I believe it would have been agreeable to the feelings of the members of this Institute if Mr. Jellicoe had continued to hold the office of President. (Hear, hear.) He had at different times expressed his wish to retire, and had reconsidered his determination at the request of the Council; but at last his feelings were so strong on the subject that the Council had nothing to do but to acquiesce in them. All I can say in reference to the subject is that, having lost such a President as Mr. Jellicoe, we are very fortunate in having such a President as Mr. Brown. (Hear, hear.) Before reading the resolution, I beg leave to offer my thanks to Mr. Brabrook, for his valuable suggestions on the subject of the *Journal*. No doubt the observations he has made will be duly considered by the members of the Council. Gentlemen, I will not detain you any longer, but will, if you please, read the resolution, which I trust will meet with the unanimous support of the meeting:—'That the members of the Institute of Actuaries desire to offer to Charles Jellicoe, Esq., upon his resignation of the office of President, their warmest thanks for his valuable services and unceasing attention to its interests, both previously to and during the time he held that office.'" (Cheers.)

Mr. BAILEY—"Mr. President, such a resolution as this needs no seconding, as it is sure to be agreed to with unanimity. I, however, claim the privilege of being the formal seconder, because I believe of all men in this room I am under the deepest possible obligation to Mr. Jellicoe. I am the only one present, I believe, who has served under Mr. Jellicoe for a period of more than 14 years in a subordinate capacity. During the whole period I never experienced from him one unkind speech or a single harsh word, even amidst all the turmoil and conflicting interests which arise in ordinary business affairs. (Hear, hear.) Mr. Hodge has done justice to the services Mr. Jellicoe has rendered to the Institute in a way it would be impossible for me to attempt; but, Sir, few of the members are aware of the extent of those services. I doubt whether they are aware of the amount (which I have had some means of observing) of time and ability—and the money—which Mr. Jellicoe has devoted to this Institute. (Hear, hear.) It is, I believe, not generally known that the early numbers of the *Assurance Magazine*—it was not then called the *Journal*—were published at Mr. Jellicoe's sole risk; and any gentleman who knows at all what sort of profit accrues from scientific publications, will know what that really implies. (Hear, and laughter.) I hope I am not intruding in making these remarks; but as Mr. Jellicoe is not present, we can say what we have done more readily than we could in his presence." (Hear, hear.)

Mr. SPRAGUE.—"Perhaps, Sir, you will allow me also, on the same ground as Mr. Bailey, to say a few words in support of this resolution. I too feel that I am under considerable personal obligation to Mr. Jellicoe; and although I was not under him for a period of 14 years, still I may say I served my apprenticeship to the profession under him. Not only did I obtain my first knowledge of the subject from him, but whenever at a subsequent period I have wished for advice, to have my opinion revised or confirmed, I have always found him most willing to assist. I am sure all who know him will say the same." (Hear, hear.)

The PRESIDENT—"If I were to allow every one to speak on this resolution who would wish to do so our proceedings would be protracted to a great extent. I feel greatly the services of Mr. Jellicoe, having been associated with him from the very beginning of the Institute. In the first year I worked with him in the editorship of the *Journal*, and during that time we acted together most cordially. Ever since that time I have had so many opportunities of seeing the great ability he has brought to bear upon this Institute, that every year has increased both my affection and esteem for him." (Applause.)

The resolution was then put and carried *nem. con.*

The scrutineers reported that the following gentlemen had been unanimously elected office-bearers for the ensuing year:—

President.

SAMUEL BROWN.

Vice Presidents.

WILLIAM BARWICK HODGE.	ROBERT TUCKER.
THOMAS BOND SPRAGUE, M.A.	J. HILL WILLIAMS.

Council.

Arthur Hutcheson Bailey.
Samuel Brown.
Charles John Bunyon, M.A.
David Chisholm.
George Cutcliffe.
Archibald Day.
Henry Devereux Davenport.
Alexander Glen Finlaison.
Alexander Pearson Fletcher.
Edwin Henry Galsworthy.
William John Hancock.
Augustus Hendriks.
William Barwick Hodge.
Charles Jellicoe.
Jeremiah Lodge, B.A.
Henry Marshal.
William Matthew Makeham.
Frank McGedy.
James Meikle.
John Messent.
Edward A. Newton, M.A.
William P. Pattison.
Henry William Porter, B.A.
John Reddish.
Henry Ambrose Smith.
William Spens.
Thomas Bond Sprague, M.A.
Robert Tucker.
John Hill Williams.
W. S. B. Woolhouse.

Treasurer.

George Cutcliffe.

Honorary Secretaries.

Arthur Hutcheson Bailey.
Archibald Day.

The President, in returning thanks, said he had always taken the deepest interest in all the concerns of this Institute, and he felt highly the compliment that had been paid him. Although he might be unable to increase the reputation of the Institute beyond that which it had attained under Mr. Jellicoe's presidency, he trusted he should be able to follow in the footsteps he had marked out, and, as far as his ability availed, to give his best services to the Institute. On behalf of the other gentlemen also, who were deeply interested in the Institute, he returned thanks for the kindness that had been paid them on this as well as on previous occasions.

A vote of thanks was given to the scrutineers.

The President then read the following address :—

" Gentlemen,—The important step taken by our late distinguished President, in resigning the office which he has so long held with the highest honour to himself, and the greatest advantage to this Institute, seems a proper occasion on which to take a brief review of the progress of the Institute and of the manner in which it has accomplished the objects for which it was founded. Though grateful for the kindness of my colleagues on the council in electing me to fill up the vacancy to the conclusion of the year of office, I was amongst the most earnest of those who desired Mr. Jellicoe to reconsider his determination, and favour us with the continuance of his eminent services in that position which, with our unanimous consent, he was so qualified to fill. But his resignation, like all his previous actions, was dictated by the purest wish to promote what he thought the real interests of the Institute. Believing that the opening up of so high an object of ambition might stimulate others to greater efforts to prove worthy of it, and feeling that public services to our profession could alone be entitled to look for so great a reward, he persevered in his generous intentions, and voluntarily laid down an honour of which no one wished to deprive him, in order that others might share it in turn, and thus a happy emulation of the common good be excited. I trust to vie with Mr. Jellicoe's public spirit, and hold the position for a brief period in trust only for others who, by greater services to the Institute, may hereafter be deemed still more worthy of the honour, high as I esteem it.

" In considering the benefits which the Institute has conferred upon its members, we must recollect the nature of our profession and the calls which it makes for something more than a mere pratical knowledge of business. The doctrine of probability—alluded to in the first section of our constitution and laws—engaged, more than two hundred years ago, the attention of the eminent mathematicians of that day; and although its practical application began in settling the chances of success between two players engaged at dice, and many of the subsequent discoveries and general principles were laid down by imagining all the possible combinations in games of chance, it excited the interest and was gradually improved by men of the highest intellect and genius. It would be very interesting to discover the original correspondence between Pascal and

Fermat on the case first put to him, in which the latter generalised and found a rule for the chances of success at any moment of interrupting the game and between any number of players. When we recall such names as Leibnitz, Huyghens, James and Nicholas Bernoulli, Remond de Montmort, De Moivre, Daniel Bernoulli, Euler, Lagrange, and Laplace abroad, we trace a long succession of the highest mathematical minds, each enlarging and improving the general theory and the application of the subject. It was James Bernoulli who intended to apply the reasoning to the events of evil government and moral and political economy. His death interrupted these important studies. The most varied application of the science was suggested by Condorcet in 1784, who endeavoured to ascertain the probability of a numerous assembly coming to a correct judgment by votes, of the justice of criminal judgments, of the correctness of tradition, and the truth of history. He left at his death a plan of reducing to mathematical laws events depending on the free will of man, in which, of late years, M. Quetelet, the Astronomer Royal of Belgium, and Perpetual Secretary of the Royal Academy of Sciences there, has made such remarkable researches.

"The application of the doctrine of chances to human life is first traced with any authority to the Grand Pensioner, De Wit, who in 1671 used it by certain hypotheses to calculate the values of life on a resolution of the States General of Holland to negotiate funds on life annuities. I think we may be proud of finding in the *Journal of the Institute* the original report of that eminent statesman, which, after the lapse of 180 years, was recovered by the acuteness and perseverance of Mr. Frederick Hendriks, to whom we are also indebted for so many other valuable contributions to the *Journal* on the history of our science.

"From an early period the doctrine of probabilities applied to human life was studied in England. Simpson applied his laws of chance to annuities in 1742. Dr. Price in 1769, in his *exposé* of the defective state of many widow and annuity schemes then becoming bankrupt, showed how such studies could be turned to the public benefit; and during this period Kerseboom in Holland, Sussmilch in Germany, and Deparcieux in France, contributed by their labours to researches which, however defective they may appear to us in our present advanced state, contributed to lay the foundations of the doctrine of life contingencies which has now taken so useful a form and so vast an expansion.

"It was not till after the establishment of the Equitable Society in 1762, on pure theoretical principles of life assurance, that the status of an actuary, as we now understand it, was recognised, although the labours of Simpson and Halley really led to the first premium tables, graduated by age, being calculated. The judicious advice of Dr. Price, and the caution and ability with which his nephew, Mr. W. Morgan, conducted the early affairs of that Society, led to that great success which encouraged the formation of other companies, both mutual and proprietary, on a scale which required in the management a knowledge beyond the mere details of business. To ascertain the true position and prospects of a company, and to fix it on a firm basis, the qualifications and studies of an actuary were essential.

"But more than eighty years passed over without any great union or cordial co-operation being established amongst those who were engaged in these pursuits. No doubt a friendly feeling existed amongst them, and they met now and then before committees of the House of Commons, and on other public occasions. Some very valuable works had also appeared during this long period worthy of the great reputation of the authors, and showing a considerable advance in the practical application of the science.

"At last it was felt that it would be a great advantage to the public, as well as to the members of a profession on which such important interests depend, that they should be united in one body, so as to benefit by each other's experience, and to keep down the pretensions of ignorant or unqualified persons. The difficulties were great, as in the early discussions there did not appear to be that unanimity as to the nature of the association which would be necessary to obtain a charter of incorporation. Some seemed to prefer a mere union of managers to settle from time to time, by common consent, such questions as may arise in the course of business, and thus prevent the diversity of practice which produced a bad effect in the opinion of the public. Others advocated the higher idea of testing by examination the competence of future aspirants to the character of an actuary, and that none but those duly qualified, and who had obtained a certificate from, and admission by the council, should be entitled to practise the profession.

"Such an incorporated Society necessarily implied that all who were then actuaries should be admitted as the original members; and it is to be regretted, I think, that the want of unity of opinion prevented its formation, as by this time nearly all the objections of those who opposed it would have vanished, and the whole profession would have been united in one, for the common safety and mutual advantage.

" But, as it is, we must congratulate ourselves that the Institute of Actuaries took the form of an intellectual, scientific, and educational body, rather than a mere association for improvements in the practice of the business. It has thus been the means of bringing out an amount of talent of which we are now beginning to reap the fruits, and which in the meantime has afforded to all concerned, whether the older or the younger members of the profession, much gratification, and many opportunities of improvement. It cannot be said that the subjects of study are not of the highest interest and the widest range. The doctrine of probability may be applied to all classes of events of which the recurrence may be constantly predicted under similar circumstances or from similar causes. The law of mortality, taking death as a certain event at some time or other, is but one of many inquiries, though one of the highest importance from the monetary interests of life assurance companies which now depend upon it. The laws of disease, of duration and intensity of sickness, are equally necessary to be known to the actuary who would desire to place on a safer foundation the thousands of Friendly Societies or Sickness Clubs so essential to the well-being and comfort of the operative classes.

" The extension of these inquiries to marine risks, by the establishment of the Statistical Committee of Lloyd's, may lead to the most valuable information on the causes of loss, under various circumstances, affecting the enormous commercial interests of this country. And there can be no reason why the probability of happening of events, and the proportion in which different causes may combine to bring them about, may not in many other cases, such as fires, bankruptcies, guarantees of fidelity, hail and cattle, and agricultural insurances, be reduced to mathematical laws by those most versed in the proper knowledge or skill in these subjects.

" The valuable researches of Mr. Day into the marriages of the peerage families have already given us some definite notions on the premiums to be charged in issue cases—a subject which, by its practical application, may make many interests in reversionary property valuable, which a few years ago would have been wholly unsaleable.

" But even if we confine ourselves at present to ascertaining the true law of mortality, what a vast field of research is opened up! The theory of Gompertz was only propounded by him after analyzing and comparing the best observations he could collect together. He concluded that if there were a number of diseases affecting young and old alike, and no other diseases, life at all ages would be of equal value, and the number living and dying would decrease in geometrical progression as age increases by equal intervals. Mr. Makeham, in his recent most valuable papers, shows, from observations in England generally, that there are certain classes of disease acting precisely in this way, thus enabling him, by his important modification, to reduce all tables to the same simple form. But it requires a much greater number of facts of this kind to be carefully analyzed in order to account for the fluctuations and divergences from the law, which will be found at different periods of life, or under different circumstances, and in various countries. For instance, in passing from birth to maturity, how much may depend on the treatment of infants by their mothers, by nurses, by medical men. What differences may be educed by climate, by habits of living, by fashion, by exercise, by the fresh air of the country, or the sanitary conditions of town life. Again, how may the natural law of mortality be affected by removing to a different climate, and having to undergo the trials to the constitution which may occur to a native of the temperate zone in becoming acclimatized in tropical countries. The general result of the latter appears to be, that the force of mortality is equal at all the ages to which an European is generally exposed to its influences.

" An essential part of this inquiry is the different kinds of disease which act upon different ages of life. In this respect the very accurate returns of the causes of death in the Gotha Life Assurance Company, which, some years ago, were analyzed by Mr. Neison, in the *Journal of the Statistical Society*, may afford to inquirers like Mr. Makeham some novel and curious results. We may be gratified to hear that Mr. Hopf, the able Manager of that Company, and one of our foreign corresponding members, is engaged in preparing a much more elaborate report of the experience of his Company for nearly 30 years; and if, as he expects, the observations will extend over 10,000 or 12,000 deaths, the scientific results on these interesting questions may prove of the highest value. Within the last few months a similar return of causes of death in the Prudential Assurance Company, by Mr. Harben, although only for three years, may prove of great value on account of the large number of lives observed, nearly 250,000 in each year, and of the class to which they principally relate, the provident part of the working classes of this country.

"I need hardly call attention under this head to the elaborate statistical reports of the Registrar-General, in which Dr. Farr has rendered great service to this country by the full and careful reports which he has given on a vast variety of topics affecting the progress of the population and the public health. I believe them to be models for similar reports in other countries.

"I have alluded to this question of the law of mortality, because, though only one of the numerous questions which have engaged the attention of the Institute during the last few years, it is of the highest importance, and suggests a great number of topics for future discussion. When we recall the valuable papers which it has been the means of eliciting from Messrs. Gompertz, Edmonds, Sprague, Peter Gray, De Morgan, Woolhouse, and Makeham, not to mention others who have incidentally alluded to it, it shows that we have no lack of matter to occupy the most ardent and philosophical inquirers or the most acute reasoners on the observations of others.

"It would take too long time to allude to all the other subjects which have excited at our sessional meetings such lively and thought-inspiring discussions. They are all of the deepest interest, and mostly bearing directly on the questions which constantly fall within an actuary's province to decide. In early days, Mr. Hardy's papers on annuities and probabilities, Mr. Hodge's on interest of money and mortality in naval and military operations; Jellicoe, Higham, Pinckard, and Spens, on original tables of observation of Life Assurance Companies; Bailey and Day on other observations, and Meikle on the arrangement of data now being collected, form but a small part of the many topics discussed. To these we have to add Hopf, Lazarus, Homans, and others, on foreign life assurance matters.

"Many papers, too, on subjects which have not been read or discussed at the public meetings, will be found in the *Journal of the Institute*. De Morgan, Farr, Finlaison, and Sir J. W. Lubbock, have contributed several papers of the highest importance; whilst the papers by Messrs. Atkins, Christie, and Miller, on questions in fire insurance, and Rucker on the statistics of marine insurance, show the variety of subjects to which attention may be directed with advantage.

"Indeed, it is impossible to deny that the publication of the *Journal*, as well as the public discussion of such questions, have been the means of calling out an amount of talent and ability, which, if not lost, might have been, at least for many years, unknown or known but to few. The reputation of the *Journal* as a medium of scientific information on our subjects is not only recognised in this country, it is frequently quoted by the leading writers in Germany, France, and the United States of America, and everywhere appealed to as worthily maintaining the great objects for which it was commenced.

"Nor have we less reason to be gratified with the results of the examinations in the Institute, which are the more honourable as they are entirely voluntary; and do not, as in the Government examinations, lead to any immediate reward, but are only evidences of the ability and praiseworthy efforts to excel, which I trust every one who has undergone them will in the end find rewarded with the success they deserve. That for 18 years these examinations should find candidates coming forward with renewed zeal to keep pace even with the higher standard which the progress of education in the present day requires, is a subject of real gratification to those who first advocated their institution, and of great congratulation to those younger members who obtain the honour of passing them. They may well rest satisfied with the reputation thus obtained until opportunities occur when their labours will be rewarded with more substantial success. It is cheering to observe that many of those who have most distinguished themselves in these examinations have already received the rewards they merited, and are doing credit to the Institute by the high estimation in which they are held by their boards of directors.

"The list of our honorary and corresponding members also indicates the character and utility of the Institute. We may be proud to find in the former such names as Mr. Babbage, Professor De Morgan, Dr. Farr, the late Mr. Gompertz, Mr. Peter Gray, Lord Overstone, Mr. Tidd Pratt, M. Quetelet, and Professor Sylvester, and in the latter almost all the talented men in every country of Europe and in America who have studied and written upon the theory of our science or reduced it to practice with a success which in some cases, especially by its recent progress in America, astonishes us with its vast results.

"I may allude to one other topic in which the Institute is likely to confer the greatest benefit on the profession and the assurance business generally, I mean the collection of the actual experience of Life Assurance Companies. Many years have elapsed since the combined experience of the 17 Life Assurance Companies was made public. The facts relating to the experience of particular Companies have been published by their actuaries. and allow of comparisons of the highest interest. The experience of the Equitable by

Mr. Arthur Morgan, which was given in a form to allow of the mortality and the effect of selection being traced for every age of entry, the Amicable by Mr. Galloway, the Eagle by Mr. Jellicoe, the Economic by Mr. Downes, the Clerical, Medical and General by Mr. Pinckard, the Scottish Amicable by Mr. Spens, the Metropolitan by Mr. Pearson, all contain tables of observation worthy of careful examination. But to bring together a large mass of observations of combined Offices, to eliminate duplicate lives, to pursue the inquiry into the older ages of assured lives, to ascertain the effect of selection and its consequences not merely on the profits of a Company in the early but in the later periods of its career, and to settle the difficult question of the relative mortality of female lives in assurances or annuities, required an agreement amongst the older Companies to put together sufficient materials for these most important inquiries. As far back as 1862 the Council of the Institute therefore agreed, with a committee of the Scotch Companies, to unite for this common object, and to collect the facts on cards which will allow of any desired arrangement. Circumstances have somewhat delayed the combination of all the materials, but it is hoped that both the Scotch and English collection is now so far advanced that in a very short time this great work will be brought to a conclusion. It seems probable that the united experience will comprise 200,000 distinct lives from their date of entry, after eliminating subsequent policies on the same life. If so, the results will be the most trustworthy, and the most valuable contributions to our science which has ever yet been made.

" Time will not permit me to review many other points of our progress which are yet of the deepest interest to our members. It is evident that since the establishment of the Institute the profession has gained a higher scientific position as a whole than it held before. There were then, as there are now, many members of it who had individually distinguished themselves, and who retained their place in their profession without the association, the emulation, and the mutual interchange of knowledge and experience which have given to our meetings a character so useful and agreeable. To all who are striving in their separate spheres to do honour to actuarial pursuits, whether in a body, as those gentlemen who constitute the Actuaries' Club in London, the Association of Managers, and the Faculty of Actuaries in Scotland, or the Congress of Life Assurance Representatives in America, or as the distinguished individuals who in Germany, France, and other countries, by their writings and practical experience, do so much credit to the same cause, we heartily wish success; and I am sure the members of this Institute will most cordially join in rendering every mutual aid which a community of feeling and of intellectual pursuits naturally inspires. We may be permitted to hope, however, that in the course of time a more thorough union will be effected, and one body comprise all the members within it.

" I will merely conclude, as I began, by a reference to the great services rendered to the Institute by our late President, Mr. Jellicoe. By his talents and ability, as President of the Institute, repeatedly and unanimously elected for seven successive years, by his ardent efforts to promote its interests, by his unceasing watchfulness to seize every occasion that could add to its lustre, by his able editorship of the *Journal* for nearly eighteen years, and by his skill in drawing out fresh talent, both in the discussions and in the *Journal*, he has raised the reputation of the Institute, both at home and abroad, to a degree which we may be excused for desiring to maintain without having much hope of increasing."

The reading of this address was frequently interrupted by expressive marks of approval.

On the motion of Mr. Hodge, seconded by Mr. Spencer, a vote of thanks was given to Mr. Brown for his able address, and the hope was expressed that it would be printed in the *Journal.*

Mr. Messent remarked that there was one change amongst the officers of the Institute which had not been noticed by any of the speakers, and that was the appointment, as one of the Hon. Secretaries, of Mr. Bailey in the room of Mr. J. Hill Williams, who had done much to advance the interests of the Institute and to obtain for it the proud position it occupied amongst kindred Societies. (Hear, hear.) He had, therefore, much pleasure in proposing that the thanks of the meeting be tendered to Mr. Williams.

Mr. Hodge seconded the resolution, which was carried unanimously.

A cordial vote of thanks to the President for presiding brought the meeting to a close.*

---

* The above report of the proceedings of the annual meeting is extracted from the *Insurance Record.*

INSTITUTE OF ACTUARIES.

Dr. *Abstract of Cash Account from 1st April, 1866, to 31st March, 1867.* Cr.

RECEIPTS.

	£	s.	d.	£	s.	d.
To Cash balance brought forward	270	15	10			
Petty cash „ „	0	18	9			
				271	14	7
To Subscriptions for 1866-67, viz.:—						
61 Fellows, Town, at £3 3 0	192	3	0			
25 „ Country, „ 2 2 0	52	10	0			
79 Associates, Town, „ 2 2 0	165	18	0			
26 „ Country, „ 1 1 0	27	6	0			
				437	17	0
Arrears	18	18	0			
Certificate Fee	5	5	0			
Composition	31	10	0			
Dividends	41	6	0			
				96	19	0
				£806	10	7

	£	s.	d.
Note.—Invested in 3 per Cent. Consols (£900 Stock)	823	19	3
Cash balance	259	2	0
Books in Library (say)	366	18	9
	£1,450	0	0

PAYMENTS.

	£	s.	d.	£	s.	d.
By Purchase of £200 3 per Cent. Consols				177	5	0
Rent	87	10	0			
Salaries	105	0	0			
Journal	66	6	0			
Stationery and sundry Printing	38	12	3			
Ordinary Meetings	23	5	1			
Fire and Light	12	8	10			
Postage and Receipt Stamps	10	10	0			
Library	8	15	7			
Advertising Examinations	5	1	0			
House and Petty Expenses	6	13	0			
Miscellaneous	6	1	10			
				370	3	7
Balance in Bank, carried forward	251	7	0			
Petty Cash balance	7	15	0			
				259	2	0
				£806	10	7

Examined and found correct, 26th April, 1867.

(Signed) B. Newbatt, }
(„) G. W. Berridge, } *Auditors.*
(„) Ralph P. Hardy, }

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